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Smoothness of the Scalar Coefficients in the Representations of Isotropic Tensor-Valued Functions

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For a three-dimensional space, an isotropic tensor-valued function $\mathbf{\Phi}$ of a symmetric tensor $\mathbf{A}$ has the representation $\mathbf{\Phi}(\mathbf{A}) = \alpha(\mathbf{A})\mathbf{I} + \beta(\mathbf{A})\mathbf{A} + \gamma(\mathbf{A})\mathbf{A}^2$, in which the coefficients α, β, and γ are isotropic scalar-valued functions. It is known that these coefficients may fail to be as smooth as $\mathbf{\Phi}$ at those tensors $\mathbf{A}$ that do not have three distinct eigenvalues. Serrin (1959) and Man (1994) determined conditions on the smoothness of $\mathbf{\Phi}$ that guarantee the existence of continuous coefficients. We give a different proof of their results and also determine conditions on $\mathbf{\Phi}$ that guarantee the existence of continuously differentiable coefficients.

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1 Introduction

Let $\mathcal{V}$ be a three-dimensional real inner product space, and let $\mathbb{R}$ denote the real numbers or scalars. Let *Sym* denote the linear space of symmetric (second-order) tensors on $\mathcal{V}$. Let $\mathcal{U}$ be a nonempty subset of *Sym* which is *invariant* in the sense that $\mathbf{A} \in \mathcal{U}$ iff $\mathbf{QAQ}^{\mathsf{T}} \in \mathcal{U}$ for each orthogonal tensor $\mathbf{Q}$. Then a *scalar-valued isotropic function* on $\mathcal{U}$ is a function $\psi : \mathcal{U} \to \mathbb{R}$ with the property $\psi(\mathbf{QAQ}^{\mathsf{T}}) = \psi(\mathbf{A})$ for each $\mathbf{A} \in \mathcal{U}$ and each orthogonal tensor $\mathbf{Q}$. A *tensor-valued isotropic function* on $\mathcal{U}$ is a function $\mathbf{\Phi} : \mathcal{U} \to Sym$ with the property $\mathbf{\Phi}(\mathbf{QAQ}^{\mathsf{T}}) = \mathbf{Q\Phi}(\mathbf{A})\mathbf{Q}^{\mathsf{T}}$ for each $\mathbf{A} \in \mathcal{U}$ and each orthogonal tensor $\mathbf{Q}$. For any symmetric tensor $\mathbf{A}$, let $^{\#}\mathbf{A} \in \{1, 2, 3\}$ denote the number of distinct eigenvalues of $\mathbf{A}$. Let

$$Sym_n := \{\mathbf{A} \in Sym : {}^{\#}\mathbf{A} = n\} \qquad (n = 1, 2, 3)\,. \tag{1.1}$$

In particular, Sym_1 is the one-dimensional subspace of all spherical tensors $a\mathbf{I}$, $a \in \mathbb{R}$. Each set Sym_n is invariant, and $Sym = Sym_1 \cup Sym_2 \cup Sym_3$. Let

$$\mathcal{U}_n := \mathcal{U} \cap Sym_n = \{\mathbf{A} \in \mathcal{U} : {}^{\#}\mathbf{A} = n\} \qquad (n = 1, 2, 3)\,. \tag{1.2}$$

Then each set $\mathcal{U}_n$ is invariant, and $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$. A scalar-valued or tensor-valued function on $\mathcal{U}$ is isotropic iff its restriction to each of the invariant subsets $\mathcal{U}_n$ is isotropic.

According to a well-known theorem of Rivlin and Ericksen [1, §29, §39], a function $\mathbf{\Phi} : \mathcal{U} \to Sym$ is isotropic iff there are scalar-valued isotropic functions α, β, γ on $\mathcal{U}$ such that

$$\mathbf{\Phi}(\mathbf{A}) = \alpha(\mathbf{A})\mathbf{I} + \beta(\mathbf{A})\mathbf{A} + \gamma(\mathbf{A})\mathbf{A}^2, \tag{1.3}$$

where $\mathbf{I}$ denotes the identity tensor. Since the publication of Rivlin and Ericksen's classic paper in 1955, various proofs of this representation theorem have appeared in the mechanics literature; cf. Serrin [2], [3, §59], Truesdell & Noll [4, §12], Wang & Truesdell [5, §III.2], Gurtin [6, §37], and Wang [7].[1] The proofs in [1]–[7] have the following features in common. The authors establish (1.3) by showing that it holds on each of the subsets $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$. In particular, they show that α, β, γ are uniquely determined by $\mathbf{\Phi}$ on $\mathcal{U}_3$. They set

$$\gamma(\mathbf{A}) = 0\,, \ \ \forall\, \mathbf{A} \in \mathcal{U}_2\,, \tag{1.4}$$

and show that there are unique functions α, β on $\mathcal{U}_2$ satisfying (1.3) and (1.4). Finally, they set

$$\beta(\mathbf{A}) = \gamma(\mathbf{A}) = 0\,, \ \ \forall\, \mathbf{A} \in \mathcal{U}_1\,, \tag{1.5}$$

and show that there is a unique function α on $\mathcal{U}_1$ satisfying (1.3) and (1.5). Isotropy of the coefficients is established in one of two ways: either by verifying indirectly that

[1] Applications of (1.3) to the response functions of isotropic materials are discussed in [1]–[6].

$\gamma(\mathbf{QAQ}^{\mathsf{T}}) = \gamma(\mathbf{A})$, for example, without actually solving for the coefficients (cf. [4]–[7]), or by deriving explicit isotropic formulas for the coefficients. The latter approach was used by Rivlin and Ericksen [1] (cf. also Serrin [2, 3]), and their formulas for the case $\#\mathbf{A} = 3$ are of interest here.

Let a_1, a_2, a_3 denote the eigenvalues of the symmetric tensor $\mathbf{A}$, let $I_{\mathbf{A}}, II_{\mathbf{A}}, III_{\mathbf{A}}$ denote the principal invariants of $\mathbf{A}$, and let $\overline{II}_{\mathbf{A}}, \overline{III}_{\mathbf{A}}$ denote the second and third moments of $\mathbf{A}$. Then (cf. Ericksen [8, §38])

$$
\begin{aligned}
I_{\mathbf{A}} &= a_1 + a_2 + a_3 = \operatorname{tr}\mathbf{A} = \mathbf{A}\cdot\mathbf{I}, &(1.6)\\
II_{\mathbf{A}} &= a_1a_2 + a_2a_3 + a_3a_1 = \tfrac{1}{2}(I_{\mathbf{A}}^2 - \overline{II}_{\mathbf{A}}), &(1.7)\\
III_{\mathbf{A}} &= a_1a_2a_3 = \det\mathbf{A} = \tfrac{1}{6}I_{\mathbf{A}}^3 - \tfrac{1}{2}I_{\mathbf{A}}\overline{II}_{\mathbf{A}} + \tfrac{1}{3}\overline{III}_{\mathbf{A}}, &(1.8)\\
\overline{II}_{\mathbf{A}} &= a_1^2 + a_2^2 + a_3^2 = \operatorname{tr}\mathbf{A}^2 = \mathbf{A}\cdot\mathbf{A} = \|\mathbf{A}\|^2, &(1.9)\\
\overline{III}_{\mathbf{A}} &= a_1^3 + a_2^3 + a_3^3 = \operatorname{tr}\mathbf{A}^3 = I_{\mathbf{A}}^3 - 3I_{\mathbf{A}}II_{\mathbf{A}} + 3III_{\mathbf{A}}. &(1.10)
\end{aligned}
$$

Here $\|\mathbf{A}\|$ denotes the norm of $\mathbf{A}$ corresponding to the inner product $\mathbf{A}\cdot\mathbf{B} = \operatorname{tr}(\mathbf{AB})$ on *Sym*. From (1.3) we obtain the following system of equations for $\alpha(\mathbf{A}), \beta(\mathbf{A}), \gamma(\mathbf{A})$:

$$
\begin{bmatrix} 3 & I_{\mathbf{A}} & \overline{II}_{\mathbf{A}} \\ I_{\mathbf{A}} & \overline{II}_{\mathbf{A}} & \overline{III}_{\mathbf{A}} \\ \overline{II}_{\mathbf{A}} & \overline{III}_{\mathbf{A}} & \operatorname{tr}\mathbf{A}^4 \end{bmatrix}
\begin{bmatrix} \alpha(\mathbf{A}) \\ \beta(\mathbf{A}) \\ \gamma(\mathbf{A}) \end{bmatrix}
=
\begin{bmatrix} \operatorname{tr}\mathbf{\Phi}(\mathbf{A}) \\ \operatorname{tr}(\mathbf{\Phi}(\mathbf{A})\mathbf{A}) \\ \operatorname{tr}(\mathbf{\Phi}(\mathbf{A})\mathbf{A}^2) \end{bmatrix}. \tag{1.11}
$$

Let $M_{\mathbf{A}}$ denote the coefficient matrix in (1.11). By the Cayley-Hamilton theorem,

$$
\mathbf{A}^3 = I_{\mathbf{A}}\mathbf{A}^2 - II_{\mathbf{A}}\mathbf{A} + III_{\mathbf{A}}\mathbf{I}. \tag{1.12}
$$

On multiplying (1.12) by $\mathbf{A}$ and taking the trace, we obtain an expression for $\operatorname{tr}\mathbf{A}^4$ which, together with (1.6)–(1.10), can be used to express $\det M_{\mathbf{A}}$ as a polynomial in either the principal moments, the principal invariants, or the eigenvalues of $\mathbf{A}$. In particular, we find that $\det M_{\mathbf{A}}$ is just the cubic discriminant $\Delta_{\mathbf{A}}$ of the characteristic polynomial of $\mathbf{A}$:

$$
\begin{aligned}
\det M_{\mathbf{A}} = \Delta_{\mathbf{A}} &= 18I_{\mathbf{A}}II_{\mathbf{A}}III_{\mathbf{A}} - 4I_{\mathbf{A}}^3III_{\mathbf{A}} + I_{\mathbf{A}}^2II_{\mathbf{A}}^2 - 4II_{\mathbf{A}}^3 - 27III_{\mathbf{A}}^2\\
&= (a_1 - a_2)^2(a_2 - a_3)^2(a_3 - a_1)^2. \qquad (1.13)
\end{aligned}
$$

Since $\Delta_{\mathbf{A}} \neq 0$ iff $\#\mathbf{A} = 3$, (1.11) has a unique solution for $\alpha(\mathbf{A}), \beta(\mathbf{A}), \gamma(\mathbf{A})$ iff $\#\mathbf{A} = 3$, in which case Cramer's rule yields the explicit formulas (Rivlin and Ericksen [1, §29])

$$
\alpha(\mathbf{A}) = \det M_{\mathbf{A}}^{(1)}/\Delta_{\mathbf{A}}, \quad \beta(\mathbf{A}) = \det M_{\mathbf{A}}^{(2)}/\Delta_{\mathbf{A}}, \quad \gamma(\mathbf{A}) = \det M_{\mathbf{A}}^{(3)}/\Delta_{\mathbf{A}}, \tag{1.14}
$$

where $M_{\mathbf{A}}^{(n)}$ denotes the matrix obtained by replacing the nth column of $M_{\mathbf{A}}$ by the column vector on the right-hand side of (1.11). Then the isotropy of α, β, γ on $\mathcal{U}_3$ follows from (1.14), the isotropy of the principal invariants and the moments, and the isotropy of $\mathbf{\Phi}$.

The definitions and results above place no restrictions on $\mathcal{U}$ other than invariance. However, since we are interested in the relationship between the smoothness[2] of $\mathbf{\Phi}$ and the smoothness of the coefficients in its representations, for the remainder of the paper we assume that the domain $\mathcal{U}$ of the tensor-valued isotropic function $\mathbf{\Phi}$ is also an open subset of Sym. Then $\mathcal{U}_3$ and $\mathcal{U}-\mathcal{U}_1=\mathcal{U}_2\cup\mathcal{U}_3$ are open since Sym_3 and $Sym-Sym_1=Sym_2\cup Sym_3$ are open. And since Sym_3 is dense in Sym, $\mathcal{U}_3$ and $\mathcal{U}_2\cup\mathcal{U}_3$ are dense in $\mathcal{U}$. Now from (1.3) it follows that *any smoothness properties shared by* α,β,γ *on an open subset* $\check{\mathcal{U}}$ *of* $\mathcal{U}$ *are also shared by* $\mathbf{\Phi}$. Conversely, since the principal invariants and the moments of $\mathbf{A}$ are smooth functions of $\mathbf{A}$, the formulas (1.14) imply that *on the open subset* $\mathcal{U}_3$ *the coefficient functions* α,β,γ *inherit any smoothness properties of* $\mathbf{\Phi}$. Thus the problem of determining the relationship between the smoothness of $\mathbf{\Phi}$ and the smoothness of the coefficients in its representation (1.3) reduces to determining the smoothness properties that α,β,γ inherit from $\mathbf{\Phi}$ at points in the set $\mathcal{U}-\mathcal{U}_3=\mathcal{U}_1\cup\mathcal{U}_2$, which is closed and nowhere dense relative to $\mathcal{U}$.[3] Rivlin and Ericksen [1, §29] obtained explicit formulas for $\alpha(\mathbf{A})$ and $\beta(\mathbf{A})$ when $\#\mathbf{A}=2$ and γ satisfies (1.4); and on $\mathcal{U}_1$ we simply have $\alpha=\frac{1}{3}\mathrm{tr}\,\mathbf{\Phi}$ when β and γ satisfy (1.5). But these formulas are of no use for the problem at hand. There are infinitely many other sets of isotropic functions α,β,γ for which the representation (1.3) holds (though all such sets necessarily coincide on $\mathcal{U}_3$). The particular set of coefficient functions satisfying (1.4) and (1.5), while convenient for the proof of (1.3), may fail to be continuous on $\mathcal{U}$ even for polynomial $\mathbf{\Phi}$.[4]

Now suppose $\mathbf{\Phi}$ is continuous, so that α,β,γ are necessarily continuous on $\mathcal{U}_3$. Since $\mathcal{U}_3$ is dense in $\mathcal{U}$, by a well-known theorem[5] of analysis, α,β,γ have unique *continuous* extensions to all of $\mathcal{U}$ iff their limits from within $\mathcal{U}_3$ exist at each point of $\mathcal{U}_1\cup\mathcal{U}_2$; and it is not hard to show that when these continuous extensions exist they are isotropic on $\mathcal{U}$ (cf. Proposition 2.1). However, in the explicit formulas (1.14) for α,β,γ on $\mathcal{U}_3$, the denominator $\Delta_{\mathbf{A}}\to 0$ as $\mathbf{A}\to\mathcal{U}_1\cup\mathcal{U}_2$, so sufficient conditions for the existence of these limits are by no means obvious.[6] This problem was first

[2]For the present discussion we do not attach any precise meaning to the term "smooth"; e.g., it could mean continuous, differentiable, or k-times continuously differentiable (denoted by C^k).

[3]If $\mathcal{U}$ lies in the set of nonsingular symmetric tensors, then on multiplying (1.12) by $\mathbf{A}^{-1}$ and using the representation (1.3), we obtain an alternate representation which is widely used in studies of nonlinear isotropic elastic solids (cf. [4]–[6]): $\mathbf{\Phi}(\mathbf{A})=\bar{\alpha}(\mathbf{A})\mathbf{I}+\bar{\beta}(\mathbf{A})\mathbf{A}+\bar{\gamma}(\mathbf{A})\mathbf{A}^{-1}$, where $\bar{\alpha}(\mathbf{A})=\alpha(\mathbf{A})-I\!I_{\mathbf{A}}\gamma(\mathbf{A})$, $\bar{\beta}(\mathbf{A})=\beta(\mathbf{A})+I_{\mathbf{A}}\gamma(\mathbf{A})$, and $\bar{\gamma}(\mathbf{A})=I\!I\!I_{\mathbf{A}}\gamma(\mathbf{A})$. It follows that α,β,γ are smooth at a point $\mathbf{A}$ iff $\bar{\alpha},\bar{\beta},\bar{\gamma}$ are smooth at $\mathbf{A}$.

[4]For example, by (1.12) we see that the isotropic function $\mathbf{\Phi}(\mathbf{A})=\mathbf{A}^3$ on $\mathcal{U}=Sym$ has the representation (1.3) with smooth coefficient functions $\alpha(\mathbf{A})=I\!I\!I_{\mathbf{A}}$, $\beta(\mathbf{A})=-I\!I_{\mathbf{A}}$, $\gamma(\mathbf{A})=I_{\mathbf{A}}$, whereas the α,β,γ satisfying (1.4) and (1.5) are discontinuous since they must agree with the preceding solution on Sym_3.

[5]Cf. (3.15.5) in Dieudonne [9].

[6]It is easily seen that continuity of $\mathbf{\Phi}$ is not sufficient. Consider the continuous isotropic function $\mathbf{\Phi}$ on Sym defined by $\mathbf{\Phi}(\mathbf{0})=\mathbf{0}$ and $\mathbf{\Phi}(\mathbf{A})=\|\mathbf{A}\|^{-1/2}(\mathbf{A}+\mathbf{A}^2)$ for $\mathbf{A}\neq\mathbf{0}$. Then $\beta(\mathbf{A})=\gamma(\mathbf{A})=\|\mathbf{A}\|^{-1/2}$ on Sym_3, and since these functions are unbounded on every neighborhood of $\mathbf{0}$, they cannot be extended continuously to Sym.

addressed by Serrin [2], who showed that the coefficients α, β, γ in the representaion (1.3) can be extended continuously from $\mathcal{U}_3$ to $\mathcal{U}$ if $\mathbf{\Phi}$ is C^3. In proving the existence of the limits of $\alpha(\mathbf{A}), \beta(\mathbf{A}), \gamma(\mathbf{A})$ as $\mathbf{A} \to \mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$ from within $\mathcal{U}_3$, he treated the cases $\mathbf{B} \in \mathcal{U}_2$ and $\mathbf{B} \in \mathcal{U}_1$ separately. These cases are referred to as double and triple coalescence, respectively. For double coalescence, his proof requires only that $\mathbf{\Phi}$ be C^1. Serrin [2] also gave an example of a C^1 isotropic function $\mathbf{\Phi}$ on *Sym* for which the coefficients cannot be extended continuously to *Sym*; his example is discussed in Section 2. There appears to have been no additional contributions to this problem until the paper by Man [10] thirty-five years later. Man remarks that Serrin's proof, particularly for the case of triple coalescence, was given only in outline. Man gives a different proof (with full details) of the continuity of α, β, γ for this case. His proof has the advantage that $\mathbf{\Phi}$ is only required to be C^2. On combining these results, we have

Theorem 1.1 (Serrin [2] and Man [10]) *If $\mathbf{\Phi}$ is C^2 on $\mathcal{U}$ (respectively, C^1 on $\mathcal{U}_2 \cup \mathcal{U}_3$), then there are unique continuous scalar-valued isotropic functions α, β, γ on $\mathcal{U}$ (resp. $\mathcal{U}_2 \cup \mathcal{U}_3$) for which the representation (1.3) holds.*

I am not aware of analogous results for the differentiability of the coefficients.[7] One of the goals of this paper is to fill this gap in the literature.

In the next section I confirm Theorem 1.1 using a method which is substantially different from that of Serrin and Man. Their proofs of the existence of limits of the coefficients at points in $\mathcal{U}_1 \cup \mathcal{U}_2$ are complicated by the fact that the denominator in the formulas for these coefficients goes to zero.[8] Instead, in Section 2 I show that on $\mathcal{U}_3$ there are simple formulas for the coefficients in terms of the first or second derivatives of $\mathbf{\Phi}$. From these formulas it follows trivially that the coefficients have

[7] Of course, it is well-known (cf. [1]–[4]) that if $\mathbf{\Phi}$ is polynomial in the sense that the components of $\mathbf{\Phi}(\mathbf{A})$ are polynomials in the components of $\mathbf{A}$, then the coefficients in (1.3) may be expressed as polynomials in the principal invariants of $\mathbf{A}$; cf. Serrin [2], [3, §60], for a simple proof. Regarding polynomial dependence of response functions, Truesdell & Noll [4, p. 61, footnote] remark "To us, assuming polynomial dependence seems not only unnecessary ...but unphysical. We see no sign that nature loves a polynomial, and polynomial dependence is not even invariant under change of strain measure." The primary application of the above result on polynomial dependence would seem to be to the approximation of tensor-valued isotropic functions. If $\mathbf{\Phi}$ is C^r then its rth-order Taylor approximation $\mathbf{\Phi}^{(r)}$ at a spherical tensor $c\mathbf{I}$ is an isotropic polynomial of degree r and hence has a representation of the form $\mathbf{\Phi}^{(r)}(\mathbf{A}) = \alpha_r(\mathbf{A})\mathbf{I} + \beta_r(\mathbf{A})\mathbf{A} + \gamma_r(\mathbf{A})\mathbf{A}^2$, where $\alpha_r(\mathbf{A})$, $\beta_r(\mathbf{A})$, $\gamma_r(\mathbf{A})$ are polynomials in the principal invariants of $\mathbf{A}$. However, this does not imply that α_r, β_r, γ_r are rth-order approximations of α, β, γ at $c\mathbf{I}$. Indeed, example (2.2) in the next section shows that for the case $r = 1$, the polynomial functions α_1, β_1, γ_1 may be identically zero even if α, β, γ take on all real values in every neighborhood of every spherical tensor.

Serrin [3, p. 235, footnote 2] states that the theorem on polynomial dependence may be generalized to analytic $\mathbf{\Phi}$, the coefficients in this case being analytic functions of the principal invariants.

[8] They worked with explicit formulas for α, β, γ on $\mathcal{U}_3$ similar to (1.14) but expressed in terms of the eigenvalues of $\mathbf{A}$ and $\mathbf{\Phi}(\mathbf{A})$. These formulas were noted by Rivlin and Ericksen [1, §29]; cf. also Truesdell & Noll [4, §48].

limits at points in $\mathcal{U}_1 \cup \mathcal{U}_2$ from within certain subsets of $\mathcal{U}_3$, and then existence of the limits along arbitarary paths in $\mathcal{U}_3$ can be inferred from the isotropy of the coefficients (cf. Proposition 2.1). The proof does not require a separate treatment of double and triple coalescence. A similar procedure is used in Section 4, where I establish

Theorem 1.2 *If $\mathbf{\Phi}$ is C^3 on $\mathcal{U}$, then the continuous isotropic coefficients α, β, γ in the representation (1.3) are C^1 on $\mathcal{U}$. If $\mathbf{\Phi}$ is C^2 (resp. C^3) on $\mathcal{U}_2 \cup \mathcal{U}_3$, then α, β, γ are C^1 (resp. C^2) on $\mathcal{U}_2 \cup \mathcal{U}_3$.*

My approach also differs from Serrin's and Man's in that I utilize the decomposition of $\mathbf{\Phi}$ into its spherical and deviatoric parts. For any symmetric tensor $\mathbf{A}$, let $\mathbf{A}^*$ denote the deviatoric part of $\mathbf{A}$:

$$\mathbf{A}^* := \mathbf{A} - \tfrac{1}{3} I_{\mathbf{A}} \mathbf{A}\,. \tag{1.15}$$

Then $\mathbf{A}$ is deviatoric if $\mathbf{A} = \mathbf{A}^*$ (equivalently, $I_{\mathbf{A}} = 0$); $\mathbf{A}$ is spherical iff $\mathbf{A}^* = \mathbf{0}$. Let Sym^* denote the subspace of Sym consisiting of all deviatoric symmetric tensors. The deviatoric part of $\mathbf{\Phi}$ is the function $\mathbf{\Phi}^* : \mathcal{U} \to Sym^*$ defined by $\mathbf{\Phi}^*(\mathbf{A}) := \mathbf{\Phi}(\mathbf{A})^*$. Then

$$\mathbf{\Phi}(\mathbf{A}) = \tfrac{1}{3}(\operatorname{tr} \mathbf{\Phi}(\mathbf{A}))\,\mathbf{I} + \mathbf{\Phi}^*(\mathbf{A})\,; \tag{1.16}$$

and since $\operatorname{tr} : Sym \to \mathbb{R}$ and $(\ )^* : Sym \to Sym^*$ are linear, the isotropic functions $\operatorname{tr}\mathbf{\Phi}$ and $\mathbf{\Phi}^*$ are both smooth iff $\mathbf{\Phi}$ is smooth. There are several advantages to using the decomposition (1.16): it simplifies some of the computations; it allows us to obtain the conclusions in the smoothness theorems under slightly weaker conditions on $\mathbf{\Phi}$, e.g., the conditions on $\mathbf{\Phi}$ in Theorem 1.1 can be replaced by the same conditions on $\mathbf{\Phi}^*$ together with the continuity of $\operatorname{tr}\mathbf{\Phi}$; it reveals that relatively stronger smoothness properties hold for some of the coefficients in other useful representations for $\mathbf{\Phi}$ and $\mathbf{\Phi}^*$ (see below); and it more clearly reveals the nature of the possible loss in smoothness of the coefficients.

By taking the deviatoric part and the trace of the representation (1.3), we obtain

$$\mathbf{\Phi}^*(\mathbf{A}) = \beta(\mathbf{A})\mathbf{A}^* + \gamma(\mathbf{A})(\mathbf{A}^2)^* \tag{1.17}$$

and

$$\operatorname{tr}\mathbf{\Phi}(\mathbf{A}) = 3\alpha(\mathbf{A}) + I_{\mathbf{A}}\beta(\mathbf{A}) + \overline{II}_{\mathbf{A}}\gamma(\mathbf{A})\,. \tag{1.18}$$

On substituting (1.17) (or (1.20) below) into (1.16) we obtain a representation for $\mathbf{\Phi}$ which is similar to (1.3) but has the property that the coefficient of $\mathbf{I}$ is as smooth as $\mathbf{\Phi}$. Although α, β, γ are not uniquely determined by $\mathbf{\Phi}$, from (1.18) we see that α is uniquely determined by $\operatorname{tr}\mathbf{\Phi}$, β, and γ and that α has any smoothness properties shared by $\operatorname{tr}\mathbf{\Phi}$, β, and γ. Therefore, it remains to establish the smoothness of β and γ at points in $\mathcal{U}_1 \cup \mathcal{U}_2$, and for this it suffices to consider the representation (1.17) for $\mathbf{\Phi}^*$. Actually, it is more advantageous to work with another representation which follows from (1.17). For any symmetric tensor $\mathbf{A}$, let $\mathbf{A}^{**}$ denote the deviatoric part of the square of the deviatoric part of $\mathbf{A}$:

$$\mathbf{A}^{**} := ((\mathbf{A}^*)^2)^* = (\mathbf{A}^*)^2 - \tfrac{1}{3}\mathit{II}_{\mathbf{A}^*}\mathbf{I} = (\mathbf{A}^2)^* - \tfrac{2}{3}I_{\mathbf{A}}\mathbf{A}^* , \tag{1.19}$$

where the expression on the right follows by squaring (1.15) and taking the deviatoric part. Then

$$\begin{aligned} \mathbf{\Phi}^*(\mathbf{A}) &= \theta(\mathbf{A})\mathbf{A}^* + \gamma(\mathbf{A})\mathbf{A}^{**} \\ &= -\tfrac{1}{3}\mathit{II}_{\mathbf{A}^*}\gamma(\mathbf{A})\mathbf{I} + \theta(\mathbf{A})\mathbf{A}^* + \gamma(\mathbf{A})(\mathbf{A}^*)^2 , \end{aligned} \tag{1.20}$$

where

$$\theta(\mathbf{A}) = \beta(\mathbf{A}) + \tfrac{2}{3}I_{\mathbf{A}}\gamma(\mathbf{A}) . \tag{1.21}$$

The representation $(1.20)_1$ is particularly useful in the study of deviatoric stress in isotropic elastic solids.[9] From (1.21) it follows that if β and γ are smooth on an open subset $\check{\mathcal{U}}$ of $\mathcal{U}$ then so is θ; in particular, on $\mathcal{U}_3$ the isotropic function θ is uniquely determined by $\mathbf{\Phi}$ and inherits any smoothness properties of $\mathbf{\Phi}$. However, it turns out that θ is generally smoother than either β or γ at points in $\mathcal{U}_1 \cup \mathcal{U}_2$. Indeed, the other main result of this paper is

Theorem 1.3 *If $\mathbf{\Phi}$ is C^1 on $\mathcal{U}$, then there is a unique continuous scalar-valued isotropic function θ on $\mathcal{U}$ for which the representations (1.20) hold. If $\mathbf{\Phi}$ is C^2 (resp. C^3) then θ is C^1 (resp. C^2).*

The continuity part of this theorem is proved in Section 2; the differentiability part is proved in Section 4. From (1.21) it follows that β is smooth if θ and γ are smooth; we use this fact and Theorem 1.3 to establish the smoothness of β in Theorems 1.1 and 1.2.

The proofs of Theorem 1.2 and the second part of Theorem 1.3 use Proposition 4.1, which gives conditions under which a C^r scalar-valued isotropic function on $\mathcal{U}_3$ can be extended to a C^r function on $\mathcal{U}$. The proof of this proposition is given in the Section 6; part of the proof uses some results due to Ball [13]. Section 3 contains additional formulas relating θ, γ, and β to the first and second derivatives of $\mathbf{\Phi}^*$ or $\mathbf{\Phi}$. Some of these results are used in Section 4. In the statements and proofs of our theorems on the smoothness of the coefficients α, β, γ, and θ, we regard these

[9] For example, if $\mathbf{V}$ is the left stretch tensor and $\mathbf{A} = \ln \mathbf{V}$ is the logarithmic strain tensor, then $\mathbf{A}^*$ and $\mathbf{A}^{**}$ are independent of the dilatational part $J = \det \mathbf{V}$ of the deformation. If $\mathbf{T} = \mathbf{\Phi}(\mathbf{A})$ is the Cauchy stress tensor, then $(1.20)_1$ expresses the deviatoric stress tensor as a linear combination of the volume-independent deviatoric strain tensors $\mathbf{A}^*$ and $\mathbf{A}^{**}$. The coefficients θ and γ may be expressed as functions of J (or $I_{\mathbf{A}} = \ln J$) and the volume-independent, invariant strain measures $\mathit{II}_{\mathbf{A}^*}$ and $\mathit{III}_{\mathbf{A}^*}$. For a hyperelastic material with strain energy ε per unit reference volume, $J\gamma(\mathbf{A}) = 3\,\partial\varepsilon/\partial\mathit{III}_{\mathbf{A}^*}$ and $J\theta(\mathbf{A}) = 2\,\partial\varepsilon/\partial\mathit{II}_{\mathbf{A}^*}$, with $\theta(\mathbf{0})$ equal to twice the shear modulus of the infinitesimal theory; cf. Scheidler [11]. An equivalent result in terms of the representation $(1.20)_2$ was obtained by Richter [12, $(4.6)_2$].

coefficients as (isotropic) functions from $\mathcal{U}$ into $\mathbf{R}$. They may also be regarded as functions of the principal invariants, the moments, or the eigenvalues. In Section 5 we address the question of whether the established smoothness of the coefficients, regarded as isotropic scalar-valued functions on $\mathcal{U}$, is inherited by the corresponding functions in these representations. The answer depends on the type of smoothness considered and on which representation is used. Here we also use some results due to Ball [13].

2 Continuity of the Coefficients

The main results of this section are Theorems 2.1–2.3, which give sufficient conditions for continuity of the coefficients α, β, γ, and θ. However, we begin with a discussion of Serrin's example mentioned in the Introduction. Define a scalar-valued isotropic function γ on *Sym* by

$$\gamma(\mathbf{A}) = 0 \text{ if } \mathbf{A}^* = \mathbf{0}\,, \quad \gamma(\mathbf{A}) = \sin(\|\mathbf{A}^*\|^{-1/2}) = \sin(\mathit{II}_{\mathbf{A}^*}^{-1/4}) \text{ if } \mathbf{A}^* \neq \mathbf{0}\,. \tag{2.1}$$

Then γ is discontinuous on the set Sym_1 of all spherical tensors, and this discontinuity is nonremovable since γ takes on all values between -1 and 1 in every neighborhood of Sym_1. Define a tensor-valued isotropic function $\mathbf{\Phi}$ on *Sym* by $\mathbf{\Phi}(\mathbf{A}) = \gamma(\mathbf{A})(\mathbf{A}^*)^2$. Then $\mathbf{\Phi}$ is C^∞ on $Sym_2 \cup Sym_3$, $\mathbf{\Phi}$ is differentiable with derivative $\mathbf{0}$ at each spherical tensor, and $D\mathbf{\Phi}(\mathbf{A}) \to \mathbf{0}$ as $\mathbf{A}^* \to \mathbf{0}$; thus $\mathbf{\Phi}$ is C^1 on *Sym*. From (1.19)–(1.21) we see that $\theta = 0$ and $\beta(\mathbf{A}) = -\frac{2}{3} I_{\mathbf{A}} \gamma(\mathbf{A})$. Then from (1.18) and the identity

$$\mathit{II}_{\mathbf{A}} - \mathit{II}_{\mathbf{A}^*} = \tfrac{1}{3} I_{\mathbf{A}}^2\,, \tag{2.2}$$

we have $\alpha(\mathbf{A}) = \frac{1}{9} I_{\mathbf{A}}^2 \gamma(\mathbf{A})$. From the uniqueness of α, β, γ on Sym_3 it follows that these coefficients cannot be extended continuously to *Sym* even though $\mathbf{\Phi}$ is C^1. To within some inessential constants, this is the example given by Serrin [2]. An even more pathological example is obtained by modifying his example as follows:

$$\gamma(\mathbf{A}) = \mathit{II}_{\mathbf{A}^*}^{-1/5} \sin(\mathit{II}_{\mathbf{A}^*}^{-1/5}) \text{ if } \mathbf{A}^* \neq \mathbf{0}\,. \tag{2.3}$$

Then $\mathbf{\Phi}$ is still C^1 on *Sym*, but the coefficients α, β, γ take on all real values in every neighborhood of Sym_1.

In the Introduction we observed that on the open subset $\mathcal{U}_3$ of $\mathcal{U}$, the coefficients α, β, γ, and θ are uniquely determined by $\mathbf{\Phi}$ and inherit any smoothness properties of $\mathbf{\Phi}$. But from the representations (1.17) and (1.20) it is reasonable to expect that *on $\mathcal{U}_3$ the coefficients β, γ, and θ are uniquely determined by $\mathbf{\Phi}^*$ and inherit any smoothness properties of $\mathbf{\Phi}^*$, regardless of whether or not* $\operatorname{tr}\mathbf{\Phi}$ *is smooth.* To see that this is indeed the case, we can proceed as in the derivation of (1.14) and either solve

(1.17) for β and γ, or solve (1.20) for θ and γ, and then use the relation (1.21). The latter approach, together with the identities

$$\operatorname{tr}(\mathbf{A}^*\mathbf{A}^{**}) = \overline{III}_{\mathbf{A}^*}, \quad \operatorname{tr}(\mathbf{A}^{**})^2 = \tfrac{1}{6}\overline{II}^2_{\mathbf{A}^*}, \tag{2.4}$$

yields

$$\theta(\mathbf{A}) = [\tfrac{1}{2}\overline{II}^2_{\mathbf{A}^*}\operatorname{tr}(\mathbf{\Phi}^*(\mathbf{A})\mathbf{A}^*) - 3\overline{III}_{\mathbf{A}^*}\operatorname{tr}(\mathbf{\Phi}^*(\mathbf{A})\mathbf{A}^{**})]/\Delta_{\mathbf{A}^*}, \tag{2.5}$$

$$\gamma(\mathbf{A}) = 3[\overline{II}_{\mathbf{A}^*}\operatorname{tr}(\mathbf{\Phi}^*(\mathbf{A})\mathbf{A}^{**}) - \overline{III}_{\mathbf{A}^*}\operatorname{tr}(\mathbf{\Phi}^*(\mathbf{A})\mathbf{A}^*)]/\Delta_{\mathbf{A}^*}, \tag{2.6}$$

for $\mathbf{A} \in \mathcal{U}_3$. Here $\Delta_{\mathbf{A}^*}$ is the cubic discriminant of the characteristic polynomial of $\mathbf{A}^*$ or, equivalently, of $\mathbf{A}$:

$$\begin{aligned}\Delta_{\mathbf{A}^*} &= \tfrac{1}{2}\overline{II}^3_{\mathbf{A}^*} - 3\overline{III}^2_{\mathbf{A}^*} = -4II^3_{\mathbf{A}^*} - 27III^2_{\mathbf{A}^*} \\ &= (a_1^* - a_2^*)^2(a_2^* - a_3^*)^2(a_3^* - a_1^*)^2 = \Delta_{\mathbf{A}},\end{aligned} \tag{2.7}$$

where a_i^* are the eigenvalues of $\mathbf{A}^*$.[10]

If $\mathbf{\Phi}^*$ is differentiable, then θ and γ are differentiable on $\mathcal{U}_3$, so by differentiating the representation $(1.20)_1$ we obtain the following formula for the derivative of $\mathbf{\Phi}^*$ at any $\mathbf{A} \in \mathcal{U}_3$:

$$\begin{aligned}D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}] &= (\nabla\theta(\mathbf{A})\cdot\mathbf{E})\,\mathbf{A}^* + (\nabla\gamma(\mathbf{A})\cdot\mathbf{E})\,\mathbf{A}^{**} \\ &\quad + \theta(\mathbf{A})\,\mathbf{E}^* + \gamma(\mathbf{A})(\mathbf{A}^*\mathbf{E}^* + \mathbf{E}^*\mathbf{A}^*)^*.\end{aligned} \tag{2.8}$$

Here the symmetric tensors $\nabla\theta(\mathbf{A})$ and $\nabla\gamma(\mathbf{A})$ are the gradients of θ and γ at $\mathbf{A}$ (e.g., $\nabla\theta(\mathbf{A})\cdot\mathbf{E} = D\theta(\mathbf{A})[\mathbf{E}]$), and we have used the relations

$$D_{\mathbf{A}}\mathbf{A}^*[\mathbf{E}] = \mathbf{E}^*, \quad D_{\mathbf{A}}\mathbf{A}^{**}[\mathbf{E}] = (\mathbf{A}^*\mathbf{E}^* + \mathbf{E}^*\mathbf{A}^*)^*. \tag{2.9}$$

Here and below, $\mathbf{E}$ and $\mathbf{F}$ denote arbitrary symmetric tensors. If $\mathbf{\Phi}^*$ is twice differentiable, then θ and γ are twice differentiable on $\mathcal{U}_3$, so by differentiating (2.8) and using (2.9), we obtain the following formula for the second derivative of $\mathbf{\Phi}^*$ on $\mathcal{U}_3$:

$$\begin{aligned}D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E},\mathbf{F}] &= D^2\theta(\mathbf{A})[\mathbf{E},\mathbf{F}]\,\mathbf{A}^* + D^2\gamma(\mathbf{A})[\mathbf{E},\mathbf{F}]\,\mathbf{A}^{**} \\ &\quad + (\nabla\theta(\mathbf{A})\cdot\mathbf{E})\,\mathbf{F}^* + (\nabla\gamma(\mathbf{A})\cdot\mathbf{E})(\mathbf{A}^*\mathbf{F}^* + \mathbf{F}^*\mathbf{A}^*)^* \\ &\quad + (\nabla\theta(\mathbf{A})\cdot\mathbf{F})\,\mathbf{E}^* + (\nabla\gamma(\mathbf{A})\cdot\mathbf{F})(\mathbf{A}^*\mathbf{E}^* + \mathbf{E}^*\mathbf{A}^*)^* \\ &\quad + \gamma(\mathbf{A})(\mathbf{F}^*\mathbf{E}^* + \mathbf{E}^*\mathbf{F}^*)^*.\end{aligned} \tag{2.10}$$

We wish to solve (2.8) for $\theta(\mathbf{A})$ and (2.10) for $\gamma(\mathbf{A})$. The next paragraph contains some algebraic results which will be used in these solutions.

[10] Equivalent formulas for θ and γ were derived by Blinowski [14] using a slightly different procedure; he also discusses several representations which follow from $(1.20)_1$. We could conceivably use (2.5) and (2.6) to determine sufficient conditions for the existence of the limits of $\theta(\mathbf{A})$ and $\gamma(\mathbf{A})$ as $\mathbf{A} \to \mathcal{U}_1 \cup \mathcal{U}_2$; however, as indicated in the Introduction, we will follow a different procedure here.

For any symmetric tensor $\mathbf{A}$, let $\mathcal{P}(\mathbf{A})$ denote the subspace of *Sym* consisting of all polynomials in $\mathbf{A}$, that is, all tensors of the form $c_0\mathbf{I} + c_1\mathbf{A} + c_2\mathbf{A}^2 + \cdots + c_M\mathbf{A}^M$ for some scalars $c_0, \ldots, c_M$. Then by the Cayley-Hamilton theorem,

$$\mathcal{P}(\mathbf{A}) = \operatorname{span}\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2\} = \operatorname{span}\{\mathbf{I}, \mathbf{A}^*, \mathbf{A}^{**}\}, \tag{2.11}$$

where the right equality follows from (1.15) and (1.19). Let $\mathcal{P}(\mathbf{A})^\perp$ denote the orthogonal complement of $\mathcal{P}(\mathbf{A})$:

$$\begin{aligned} \mathcal{P}(\mathbf{A})^\perp &= \{\mathbf{H} \in Sym : \mathbf{I}\cdot\mathbf{H} = \mathbf{A}\cdot\mathbf{H} = \mathbf{A}^2\cdot\mathbf{H} = 0\} \\ &= \{\mathbf{H} \in Sym : \mathbf{I}\cdot\mathbf{H} = \mathbf{A}^*\cdot\mathbf{H} = \mathbf{A}^{**}\cdot\mathbf{H} = 0\}. \end{aligned} \tag{2.12}$$

In particular, since $\mathbf{I}\cdot\mathbf{H} = \operatorname{tr}\mathbf{H}$, every $\mathbf{H} \in \mathcal{P}(\mathbf{A})^\perp$ is deviatoric, i.e., $\mathbf{H}^* = \mathbf{H}$. The dimension of $\mathcal{P}(\mathbf{A})$ is $\#\mathbf{A}$, so $\mathcal{P}(\mathbf{A})^\perp$ has dimension $6 - \#\mathbf{A}$. In particular,

$$\mathcal{P}(\mathbf{A}) = Sym_1 \quad \text{iff} \quad \mathcal{P}(\mathbf{A})^\perp = Sym^* \quad \text{iff} \quad \#\mathbf{A} = 1. \tag{2.13}$$

For any orthonormal basis $\{\mathbf{e}_i\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathcal{V}$, let $S\{\mathbf{e}_i\}$ denote the three-dimensional subspace of *Sym* consisiting of all symmetric tensors with principal basis $\{\mathbf{e}_i\}$:

$$\mathbf{A} \in S\{\mathbf{e}_i\} \;\text{ iff }\; \mathbf{A} = \sum_{i=1}^{3} a_i \mathbf{e}_i \otimes \mathbf{e}_i. \tag{2.14}$$

For any symmetric tensor $\mathbf{E}$ let $E_{ij} = \mathbf{e}_i\cdot\mathbf{E}\mathbf{e}_j = E_{ji}$ denote the components of $\mathbf{E}$ relative to $\{\mathbf{e}_i\}$, where "$\cdot$" also denotes the inner product on $\mathcal{V}$. Since a tensor $\mathbf{H}$ is orthogonal to every $\mathbf{A} \in S\{\mathbf{e}_i\}$ iff $0 = \mathbf{H}\cdot(\mathbf{e}_i \otimes \mathbf{e}_i) = H_{ii}$ for each i, the orthogonal complement of $S\{\mathbf{e}_i\}$ is the three-dimensional subspace of symmetric tensors whose diagonal components relative to $\{\mathbf{e}_i\}$ are zero:

$$S\{\mathbf{e}_i\}^\perp = \{\mathbf{H} \in Sym : \mathbf{e}_i\cdot\mathbf{H}\mathbf{e}_i = H_{ii} = 0,\; i = 1,2,3\}. \tag{2.15}$$

In particular, each $\mathbf{H} \in S\{\mathbf{e}_i\}^\perp$ is deviatoric. If $\mathbf{A} \in S\{\mathbf{e}_i\}$ then

$$\mathcal{P}(\mathbf{A}) \subset S\{\mathbf{e}_i\} \quad \text{and} \quad S\{\mathbf{e}_i\}^\perp \subset \mathcal{P}(\mathbf{A})^\perp \subset Sym^*, \tag{2.16}$$

with

$$\mathcal{P}(\mathbf{A}) = S\{\mathbf{e}_i\} \quad \text{iff} \quad S\{\mathbf{e}_i\}^\perp = \mathcal{P}(\mathbf{A})^\perp \quad \text{iff} \quad \#\mathbf{A} = 3. \tag{2.17}$$

Also, for any symmetric tensor $\mathbf{A}$ we have

$$\begin{aligned} \mathcal{P}(\mathbf{A})^\perp &= \operatorname{span}\{S\{\mathbf{e}_i\}^\perp : \{\mathbf{e}_i\} \text{ is a principal basis for } \mathbf{A}\} \\ &= \operatorname{span}\{\mathbf{H} : \mathbf{e}_i\cdot\mathbf{H}\mathbf{e}_i = 0 \text{ for some principal basis } \{\mathbf{e}_i\} \text{ of } \mathbf{A}\}. \end{aligned} \tag{2.18}$$

The following identities will be particularly useful:

$$\begin{aligned} \mathbf{E}^*\cdot\mathbf{H} = \mathbf{E}\cdot\mathbf{H} &= \operatorname{tr}(\mathbf{E}\mathbf{H}), & \forall\,\mathbf{H} \in Sym^* \\ &= 2\sum_{i<j} E_{ij}H_{ij}, & \forall\,\mathbf{H} \in S\{\mathbf{e}_i\}^\perp, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned}(\mathbf{A}^*\mathbf{E}^* + \mathbf{E}^*\mathbf{A}^*)^*\cdot\mathbf{H} &= (\mathbf{A}^*\mathbf{E} + \mathbf{E}\mathbf{A}^*)\cdot\mathbf{H} = 2\,\mathrm{tr}(\mathbf{A}^*\mathbf{E}\mathbf{H}), \qquad \forall\mathbf{H}\in\mathcal{P}(\mathbf{A})^\perp \\ &= -(a_1^* E_{23}H_{23} + a_2^* E_{31}H_{31} + a_3^* E_{12}H_{12}), \quad \forall\mathbf{A}\in S\{\mathbf{e}_i\},\ \forall\mathbf{H}\in S\{\mathbf{e}_i\}^\perp.\end{aligned} \tag{2.20}$$

In deriving $(2.20)_3$ we have used the fact that the eigenvalues a_i^* of $\mathbf{A}^*$ satisfy

$$\mathrm{tr}\,\mathbf{A}^* = a_1^* + a_2^* + a_3^* = 0. \tag{2.21}$$

If $\mathbf{\Phi}^*$ is differentiable, then from (2.8), (2.12), (2.19), and (2.20), we see that for any $\mathbf{A}\in\mathcal{U}_3$,

$$D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}]\cdot\mathbf{H} = \mathrm{tr}(\mathbf{E}\mathbf{H})\theta(\mathbf{A}) + 2\,\mathrm{tr}(\mathbf{A}^*\mathbf{E}\mathbf{H})\gamma(\mathbf{A}), \quad \forall\mathbf{H}\in\mathcal{P}(\mathbf{A})^\perp. \tag{2.22}$$

If ψ is an isotropic scalar-valued function on $\mathcal{U}$ and if $\check{\mathcal{U}}$ denotes the set of all points in $\mathcal{U}$ at which ψ is differentiable, then $\check{\mathcal{U}}$ is invariant and $\nabla\psi$ is an isotropic tensor-valued function on $\check{\mathcal{U}}$. Now from the representation (1.3) we know that $\mathbf{\Psi}(\mathbf{A})\in\mathcal{P}(\mathbf{A})$ for any isotropic tensor-valued function $\mathbf{\Psi}$, so by taking $\mathbf{\Psi} = \nabla\psi$ we obtain

$$\nabla\psi(\mathbf{A})\cdot\mathbf{K} = 0, \ \ \forall\mathbf{K}\in\mathcal{P}(\mathbf{A})^\perp. \tag{2.23}$$

If $\mathbf{\Phi}^*$ is twice differentiable, then from (2.10) with $\mathbf{F}=\mathbf{K}$, (2.12), (2.19), (2.20), and (2.23), we see that for any $\mathbf{A}\in\mathcal{U}_3$,

$$\begin{aligned}D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E},\mathbf{K}]\cdot\mathbf{H} \;=\;& \mathrm{tr}(\mathbf{K}\mathbf{H})(\nabla\theta(\mathbf{A})\cdot\mathbf{E}) + 2\,\mathrm{tr}(\mathbf{A}^*\mathbf{K}\mathbf{H})(\nabla\gamma(\mathbf{A})\cdot\mathbf{E}) \\ &+ 2\,\mathrm{tr}(\mathbf{E}^*\mathbf{K}\mathbf{H})\gamma(\mathbf{A}), \quad \forall\mathbf{H},\mathbf{K}\in\mathcal{P}(\mathbf{A})^\perp.\end{aligned} \tag{2.24}$$

And from (2.24) and (2.23), for any $\mathbf{A}\in\mathcal{U}_3$ we have

$$D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{L},\mathbf{K}]\cdot\mathbf{H} = 2\,\mathrm{tr}(\mathbf{L}\mathbf{K}\mathbf{H})\gamma(\mathbf{A}), \quad \forall\mathbf{H},\mathbf{K},\mathbf{L}\in\mathcal{P}(\mathbf{A})^\perp. \tag{2.25}$$

Let $\mathcal{H}\{\mathbf{e}_i\}$ denote the subset of $S\{\mathbf{e}_i\}^\perp$ consisting of those tensors whose off-diagonal components relative to $\{\mathbf{e}_i\}$ have absolute value 1:

$$\mathcal{H}\{\mathbf{e}_i\} := \{\mathbf{H}\in Sym : H_{ii} = 0 \text{ and } H_{ij} = \pm 1 \text{ if } i\neq j\} \subset S\{\mathbf{e}_i\}^\perp. \tag{2.26}$$

Then from (2.19)–(2.21) with $\mathbf{E}=\mathbf{H}\in\mathcal{H}\{\mathbf{e}_i\}$, we have

$$\mathrm{tr}\,\mathbf{H}^2 = 6, \quad \mathrm{tr}(\mathbf{A}^*\mathbf{H}^2) = 0, \quad \forall\mathbf{A}\in S\{\mathbf{e}_i\},\ \forall\mathbf{H}\in\mathcal{H}\{\mathbf{e}_i\}. \tag{2.27}$$

On setting $\mathbf{E}=\mathbf{H}\in\mathcal{H}\{\mathbf{e}_i\}$ in (2.22) and using (2.27), we obtain the following simple formula for θ on $\mathcal{U}_3$:

$$\theta(\mathbf{A}) = \tfrac{1}{6}D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}]\cdot\mathbf{H}, \quad \forall\mathbf{H}\in\mathcal{H}\{\mathbf{e}_i\}, \tag{2.28}$$

where $\{\mathbf{e}_i\}$ is any principal basis for $\mathbf{A}$. Similarly,

$$\operatorname{tr}\mathbf{H}^3 = 6\,\varepsilon_{\mathbf{H}}\,, \quad \varepsilon_{\mathbf{H}} := H_{12}H_{23}H_{31}\,, \ \ \forall\,\mathbf{H}\in S\{\mathbf{e}_i\}^{\perp}\,, \tag{2.29}$$

so on setting $\mathbf{L}=\mathbf{K}=\mathbf{H}\in\mathcal{H}\{\mathbf{e}_i\}$ in (2.25) and using (2.29) and (2.26), we obtain the following simple formula for γ on $\mathcal{U}_3$:

$$\gamma(\mathbf{A}) = \frac{\varepsilon_{\mathbf{H}}}{12}D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H},\mathbf{H}]\cdot\mathbf{H}\,, \quad \varepsilon_{\mathbf{H}} = \pm 1\,, \ \ \forall\,\mathbf{H}\in\mathcal{H}\{\mathbf{e}_i\}\,, \tag{2.30}$$

where $\{\mathbf{e}_i\}$ is any principal basis for $\mathbf{A}$.

If $\mathbf{\Phi}^*$ is C^1 then $D\mathbf{\Phi}^*$ is continuous, and we can use (2.28) to show that $\theta(\mathbf{A})$ has a limit as $\mathbf{A}\to\mathbf{B}\in\mathcal{U}_1\cup\mathcal{U}_2$ from within $\mathcal{U}_3$. The only difficulty is that as $\mathbf{A}\to\mathbf{B}$ along some arbitrary path in $\mathcal{U}_3$, the principal axes of $\mathbf{A}$ will generally change; in particular, they might not have limits at $\mathbf{B}$.[11] Since $\mathbf{H}$ in the formula (2.28) belongs to the set $\mathcal{H}\{\mathbf{e}_i\}$, which changes with the principal basis $\{\mathbf{e}_i\}$ of $\mathbf{A}$, it is not obvious that (2.28) has a limit at each point $\mathbf{B}\in\mathcal{U}_1\cup\mathcal{U}_2$. Similar comments apply to the formula (2.30) for γ when $\mathbf{\Phi}^*$ is C^2. Fortunately, this difficulty is easily overcome by utilizing the isotropy of θ and γ, as we now show.

For any orthonormal basis $\{\mathbf{e}_i\}$ for $\mathcal{V}$, the set $\mathcal{U}_3\cap S\{\mathbf{e}_i\}$ consists of all tensors in $\mathcal{U}$ with three distinct eigenvalues and principal basis $\{\mathbf{e}_i\}$. Included in this set is the set $\mathcal{U}_3^<\{\mathbf{e}_i\}$ consisting of all $\mathbf{A}=\sum_{i=1}^3 a_i\mathbf{e}_i\otimes\mathbf{e}_i\in\mathcal{U}$ for which $a_1<a_2<a_3$. The invariance of $\mathcal{U}_3$ guarantees that $\mathcal{U}_3\cap S\{\mathbf{e}_i\}$ and $\mathcal{U}_3^<\{\mathbf{e}_i\}$ are nonempty. And since $\mathcal{U}_3$ is an open subset of the six-dimensional space Sym, $\mathcal{U}_3\cap S\{\mathbf{e}_i\}$ and $\mathcal{U}_3^<\{\mathbf{e}_i\}$ are open subsets of the three-dimensional subspace $S\{\mathbf{e}_i\}$.

Proposition 2.1 *For a continuous scalar-valued isotropic function ψ on $\mathcal{U}_3$, the following conditions are equivalent:*

(1) ψ has a continuous extension $\bar{\psi}$ to $\mathcal{U}$;

(2) $\forall\,\mathbf{B}\in\mathcal{U}_1\cup\mathcal{U}_2$, the $\lim\limits_{\substack{\mathbf{A}\to\mathbf{B}\\ \mathbf{A}\in\mathcal{U}_3}}\psi(\mathbf{A})$ exists;

(3) $\forall\,\mathbf{B}\in\mathcal{U}_1\cup\mathcal{U}_2$ and each principal basis $\{\mathbf{e}_i\}$ for $\mathbf{B}$, the $\lim\limits_{\substack{\mathbf{A}\to\mathbf{B}\\ \mathbf{A}\in\mathcal{U}_3\cap S\{\mathbf{e}_i\}}}\psi(\mathbf{A})$ exists;

(4) $\forall\,\mathbf{B}\in\mathcal{U}_1\cup\mathcal{U}_2$ and some principal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ for $\mathbf{B}$ corresponding to the ordered eigenvalues $b_1\le b_2\le b_3$ of $\mathbf{B}$, the $\lim\limits_{\substack{\mathbf{A}\to\mathbf{B}\\ \mathbf{A}\in\mathcal{U}_3^<\{\mathbf{e}_i\}}}\psi(\mathbf{A})$ exists.

When these conditions hold, $\bar{\psi}$ is unique and isotropic on $\mathcal{U}$.

[11] Cf. the example in Kato [15, p. 128] or the two examples in Scheidler [16].

As discussed in the Introduction, the equivalence of (1) and (2) follows from the fact that $\mathcal{U}_3$ is dense in $\mathcal{U}$, and when these conditions hold $\bar{\psi}$ is unique: $\bar{\psi}(\mathbf{B}) = \lim_{\mathbf{A}\to\mathbf{B}} \psi(\mathbf{A})$, $\mathbf{A} \in \mathcal{U}_3$, for each $\mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$. Clearly, (2) $\Rightarrow$ (3) $\Rightarrow$ (4), so it remains to show that (4) implies (2) and that the continuous extension $\bar{\psi}$ is isotropic. The proof will be completed below,[12] but first we apply Proposition 2.1 to the continuity of θ and γ. Condition (3) will suffice for all the applications in this paper. However, the weaker condition (4) arises naturally in the proof below and has other applications that are not discussed here.

Let $\mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$ and let $\{\mathbf{e}_i\}$ be any principal basis for $\mathbf{B}$. Choose any $\mathbf{H} \in \mathcal{H}\{\mathbf{e}_i\}$; then the formula for $\theta(\mathbf{A})$ in (2.28) holds for each $\mathbf{A} \in \mathcal{U}_3 \cap S\{\mathbf{e}_i\}$. If $\mathbf{\Phi}^*$ is C^1, then $D\mathbf{\Phi}^*$ is continuous, so the right-hand side of (2.28) has a limit, namely $\frac{1}{6}D\mathbf{\Phi}^*(\mathbf{B})[\mathbf{H}]\cdot\mathbf{H}$, as $\mathbf{A} \to \mathbf{B}$ from within $\mathcal{U}_3 \cap S\{\mathbf{e}_i\}$, since $\mathbf{H}$ is fixed. Thus condition (3) of Proposition 2.1 holds for $\psi = \theta$, and we know that θ is continuous and isotropic on $\mathcal{U}_3$, so by Proposition 2.1 we conclude that θ has a unique continuous extension to $\mathcal{U}$. Summarizing (and replacing $\mathbf{B}$ by $\mathbf{A}$ in the above limit), we have

Theorem 2.1 *If $\mathbf{\Phi}^*$ is C^1 on $\mathcal{U}$, then there is a unique continuous scalar-valued isotropic function θ on $\mathcal{U}$ for which the representations (1.20) hold. This coefficient θ is given by (2.28) for any $\mathbf{A} \in \mathcal{U}$ and any principal basis $\{\mathbf{e}_i\}$ for $\mathbf{A}$.*

By a similar argument applied to (2.30), we conclude that γ has a continuous extension to $\mathcal{U}$ when $\mathbf{\Phi}^*$ is C^2. On combining this result with Theorem 2.1 and using the relations (1.18) and (1.21), we obtain

Theorem 2.2 *If $\mathbf{\Phi}^*$ is C^2 on $\mathcal{U}$, then there are unique continuous scalar-valued isotropic functions β, γ, θ on $\mathcal{U}$ for which the representations (1.3), (1.17), and (1.20) hold. The coefficients γ and θ are given by (2.30) and (2.28), respectively, for any $\mathbf{A} \in \mathcal{U}$ and any principal basis $\{\mathbf{e}_i\}$ for $\mathbf{A}$; then β can be expressed in terms of θ and γ by (1.21). If, in addition, $\operatorname{tr}\mathbf{\Phi}$ is continuous, then there is a unique continuous scalar-valued isotropic function α on $\mathcal{U}$ for which the representation (1.3) holds, and α is given in terms of $\operatorname{tr}\mathbf{\Phi}$, β, and γ by (1.18).*

Proof of Proposition 2.1: Assume (4) holds, so that $\mathbf{B} = \sum_{i=1}^3 b_i \mathbf{e}_i \otimes \mathbf{e}_i$ with $b_1 \le b_2 \le b_3$. Let $l_\mathbf{B}$ denote the limit in (4). Let $\{\mathbf{A}_n\}$ $(n = 1, 2, \ldots)$ be any sequence in $\mathcal{U}_3$ converging to $\mathbf{B}$, and let $\sum_{i=1}^3 a_{i,n}\mathbf{e}_{i,n} \otimes \mathbf{e}_{i,n}$, with $a_{1,n} < a_{2,n} < a_{3,n}$, be a spectral decomposition of $\mathbf{A}_n$; then $a_{i,n} \to b_i$.[13] Let $\mathbf{Q}_n$ be the orthogonal tensor which maps $\mathbf{e}_{i,n}$ to $\mathbf{e}_i$ $(i = 1, 2, 3)$. Then $\mathbf{Q}_n\mathbf{A}_n\mathbf{Q}_n^{\top} = \sum_{i=1}^3 a_{i,n}\mathbf{e}_i \otimes \mathbf{e}_i \in \mathcal{U}_3^{<}\{\mathbf{e}_i\}$, and since $a_{i,n} \to b_i$ it follows that $\mathbf{Q}_n\mathbf{A}_n\mathbf{Q}_n^{\top} \to \mathbf{B}$. Thus (4) implies that $\psi(\mathbf{Q}_n\mathbf{A}_n\mathbf{Q}_n^{\top}) \to l_\mathbf{B}$. But then

[12] Proposition 2.1 also follows from Proposition 6.1. We have given a separate proof of Proposition 2.1 in this section to avoid the discussion of certain preliminary results which are only needed for the more general Proposition 6.1.

[13] Here we have used the well-known fact that the ordered eigenvalues of a symmetric tensor are continuous functions on *Sym*. In fact, they are Lipschitz continuous; a proof can be found in Ball [13, Lemma 5.8].

$\psi(\mathbf{A}_n) \to l_{\mathbf{B}}$ since $\psi(\mathbf{A}_n) = \psi(\mathbf{Q}_n\mathbf{A}_n\mathbf{Q}_n^{\mathsf{T}})$ by the isotropy of ψ. Since $\{\mathbf{A}_n\}$ was an arbitrary sequence in $\mathcal{U}_3$ converging to $\mathbf{B}$, the $\lim_{\mathbf{A}\to\mathbf{B}} \psi(\mathbf{A})$, $\mathbf{A} \in \mathcal{U}_3$, exists and equals $l_{\mathbf{B}}$. In particular, (2) holds.

Since the continuous extension $\bar{\psi}$ coincides with the isotropic function ψ on $\mathcal{U}_3$, to prove the isotropy of $\bar{\psi}$ on $\mathcal{U}$ it suffices to show that $\bar{\psi}$ is isotropic on $\mathcal{U}_1 \cup \mathcal{U}_2$. Let $\mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$, let $\mathbf{Q}$ be any orthogonal tensor, and let $\{\mathbf{B}_n\}$ be any sequence in $\mathcal{U}_3$ converging to $\mathbf{QBQ}^{\mathsf{T}}$. Then $\bar{\psi}(\mathbf{B}_n) \to \bar{\psi}(\mathbf{QBQ}^{\mathsf{T}})$, but also $\bar{\psi}(\mathbf{B}_n) = \bar{\psi}(\mathbf{Q}^{\mathsf{T}}\mathbf{B}_n\mathbf{Q}) \to \bar{\psi}(\mathbf{B})$, since $\mathbf{Q}^{\mathsf{T}}\mathbf{B}_n\mathbf{Q} \to \mathbf{B}$. Thus $\bar{\psi}(\mathbf{QBQ}^{\mathsf{T}}) = \bar{\psi}(\mathbf{B})$. □

If $\mathbf{\Phi}$ is C^k then so are $\mathbf{\Phi}^*$ and $\operatorname{tr}\mathbf{\Phi}$. Therefore, from Theorem 2.1 we recover the first statement in Theorem 1.3, and from Theorem 2.2 we recover the case where $\mathbf{\Phi}$ is C^2 on $\mathcal{U}$ in Theorem 1.1.[14] The next theorem implies the case where $\mathbf{\Phi}$ is C^1 on $\mathcal{U}_2 \cup \mathcal{U}_3$ in Theorem 1.1. Note that we cannot use the formula (2.30) for γ in this case since we are not assuming that $\mathbf{\Phi}^*$ is twice differentiable. Let $\mathbf{B} = \sum_{i=1}^3 b_i\mathbf{e}_i \otimes \mathbf{e}_i$ be any spectral decomposition of $\mathbf{B} \in \mathcal{U}_2$. Then $b_l \neq b_m = b_n$ for some permutation l, m, n of $1, 2, 3$. Let $\mathbf{H}, \mathbf{K} \in \mathcal{S}\{\mathbf{e}_i\}^\perp$ satisfy

$$K_{lm} = 0\,, \quad K_{ln} = H_{ln} = \pm 1\,, \quad -K_{mn} = H_{mn} = \pm 1\,, \tag{2.31}$$

with H_{lm} arbitrary. Then from (2.19) and (2.20), we have

$$\operatorname{tr}(\mathbf{KH}) = 0 \quad \text{and} \quad \operatorname{tr}(\mathbf{A}^*\mathbf{KH}) = a_l^* - a_m^* = a_l - a_m\,, \ \forall\, \mathbf{A} \in \mathcal{S}\{\mathbf{e}_i\}\,. \tag{2.32}$$

On setting $\mathbf{E} = \mathbf{K}$ in (2.22), we find that for any $\mathbf{A} \in \mathcal{U}_3 \cap \mathcal{S}\{\mathbf{e}_i\}$,

$$\gamma(\mathbf{A}) = \frac{D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{K}] \cdot \mathbf{H}}{2(a_l - a_m)}\,, \quad \forall\, \mathbf{H}, \mathbf{K} \in \mathcal{S}\{\mathbf{e}_i\}^\perp \text{ satisfying (2.31)}\,. \tag{2.33}$$

Now as $\mathbf{A} \to \mathbf{B}$ from within $\mathcal{U}_3 \cap \mathcal{S}\{\mathbf{e}_i\}$, $a_l = \mathbf{e}_l \cdot \mathbf{A}\mathbf{e}_l \to \mathbf{e}_l \cdot \mathbf{B}\mathbf{e}_l = b_l$; similarly, $a_m \to b_m$. Since $b_l - b_m \neq 0$, and since $\mathbf{H}$ and $\mathbf{K}$ may be held fixed during this limiting process, we see that $\lim_{\mathbf{A}\to\mathbf{B}} \gamma(\mathbf{A})$, $\mathbf{A} \in \mathcal{U}_3 \cap \mathcal{S}\{\mathbf{e}_i\}$, exists when $D\mathbf{\Phi}^*$ is continuous. And we know that γ is continuous and isotropic on $\mathcal{U}_3$. By applying Proposition 2.1 with $\psi = \gamma$ and $\mathcal{U}$ replaced with $\check{\mathcal{U}} := \mathcal{U} - \mathcal{U}_1 = \mathcal{U}_2 \cup \mathcal{U}_3$ (so that $\check{\mathcal{U}}_1 = \emptyset$, $\check{\mathcal{U}}_2 = \mathcal{U}_2$, $\check{\mathcal{U}}_3 = \mathcal{U}_3$), we see that γ has a continuous extension to $\mathcal{U}_2 \cup \mathcal{U}_3$. On combining this result with Theorem 2.1 and replacing $\mathbf{B}$ by $\mathbf{A}$ in the above limit, we obtain

[14] The weaker conditions on $\mathbf{\Phi}$ in Theorems 2.1 and 2.2 (i.e., the requirement that $\mathbf{\Phi}^*$ is C^k while $\operatorname{tr}\mathbf{\Phi}$ need only be continuous) may have applications to isotropic elastic solids. Suppose, for example, that the material has suffered damage in the form of a random distribution of microcracks. Assuming that the damaged material can also be modeled as an isotropic elastic solid, one expects a lower bulk modulus in tension (where cracks are opening), while in compression (where all cracks are closed) the bulk modulus should ideally be the same as in the undamaged material; cf. Horii and Nemat-Nasser [17], where an analysis based on the linear theory also predicts that the shear modulus in the damaged material increases smoothly in going from tension to compression. Thus if $\mathbf{\Phi}$ is the response function for the Cauchy stress tensor in the damaged material, and hence $-\frac{1}{3}\operatorname{tr}\mathbf{\Phi}$ and $\mathbf{\Phi}^*$ the response functions for the pressure and deviatoric stress, respectively, it might be reasonable to regard $\operatorname{tr}\mathbf{\Phi}$ as continuous but with a discontinuous derivative, and $\mathbf{\Phi}^*$ as C^k $(k \geq 1)$.

Theorem 2.3 *If $\mathbf{\Phi}^*$ is C^1 on the open set $\mathcal{U} - \mathcal{U}_1 = \mathcal{U}_2 \cup \mathcal{U}_3$, then there are unique continuous scalar-valued isotropic functions β, γ, θ on $\mathcal{U}_2 \cup \mathcal{U}_3$ for which the representations (1.3), (1.17), and (1.20) hold. The coefficients γ and θ are given by (2.33) and (2.28), respectively, for any $\mathbf{A} \in \mathcal{U}_2 \cup \mathcal{U}_3$ and any principal basis $\{\mathbf{e}_i\}$ for $\mathbf{A}$; in particular, the a_l and a_m in (2.33) are the distinct eigenvalues of $\mathbf{A}$. Then β can be expressed in terms of θ and γ by (1.21). If, in addition, $\operatorname{tr}\mathbf{\Phi}$ is continuous on $\mathcal{U}_2 \cup \mathcal{U}_3$, then there is a unique continuous scalar-valued isotropic function α on $\mathcal{U}_2 \cup \mathcal{U}_3$ for which the representation (1.3) holds, and α is given in terms of $\operatorname{tr}\mathbf{\Phi}$, β, and γ by (1.18).*

3 Alternate Formulas

Here we consider alternate formulas for θ and γ and for the derivatives of $\mathbf{\Phi}$ and $\mathbf{\Phi}^*$. Some of these results are used in the next section. Throughout this section, $\mathbf{E}, \mathbf{F}, \mathbf{G}$ denote arbitrary symmetric tensors, and l, m, n denotes an arbitrary permutation of $1, 2, 3$. For any orthonormal basis $\{\mathbf{e}_i\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathcal{V}$, let

$$\mathbf{H}_l := \mathbf{e}_m \otimes \mathbf{e}_n + \mathbf{e}_n \otimes \mathbf{e}_m \in \mathcal{S}\{\mathbf{e}_i\}^{\perp}\,. \tag{3.1}$$

Then

$$\mathbf{H}_l \cdot \mathbf{H}_l = 2\,, \quad \mathbf{H}_l \cdot \mathbf{H}_m = 0\,; \tag{3.2}$$

hence $\{\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3\}$ is an orthogonal basis for $\mathcal{S}\{\mathbf{e}_i\}^{\perp}$. Also,

$$\mathbf{H}_l\mathbf{H}_m + \mathbf{H}_m\mathbf{H}_l = \mathbf{H}_n\,, \quad \operatorname{tr}(\mathbf{H}_l\mathbf{H}_m\mathbf{H}_n) = 1\,, \quad \operatorname{tr}(\mathbf{H}_l\mathbf{H}_m^2) = 0\,. \tag{3.3}$$

Since the eigenvalues a_i^* of $\mathbf{A}^*$ satisfy (2.21), for any $\mathbf{A} \in \mathcal{S}\{\mathbf{e}_i\}$ we have

$$\mathbf{A}^*\mathbf{H}_l + \mathbf{H}_l\mathbf{A}^* = -a_l^*\mathbf{H}_l\,, \tag{3.4}$$

$$(\mathbf{A}^*\mathbf{H}_l + \mathbf{H}_l\mathbf{A}^*) \cdot \mathbf{H}_l = 2\operatorname{tr}(\mathbf{A}^*\mathbf{H}_l^2) = -2a_l^*\,, \tag{3.5}$$

and

$$(\mathbf{A}^*\mathbf{H}_l + \mathbf{H}_l\mathbf{A}^*) \cdot \mathbf{H}_m = 2\operatorname{tr}(\mathbf{A}^*\mathbf{H}_l\mathbf{H}_m) = 0\,. \tag{3.6}$$

Referring to the definition (2.26) of $\mathcal{H}\{\mathbf{e}_i\}$, we see that

$$\mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_3 \in \mathcal{H}\{\mathbf{e}_i\}\,. \tag{3.7}$$

Assume $\mathbf{\Phi}^*$ is differentiable, and let $\mathbf{A} \in \mathcal{U}_3$. From (2.8) and (2.23) we obtain

$$D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}] = \theta(\mathbf{A})\,\mathbf{H} + \gamma(\mathbf{A})(\mathbf{A}^*\mathbf{H} + \mathbf{H}\mathbf{A}^*)\,, \quad \forall\,\mathbf{H} \in \mathcal{P}(\mathbf{A})^{\perp}, \tag{3.8}$$

where we have used the fact that $\mathbf{H}$ and $\mathbf{A}^*\mathbf{H} + \mathbf{H}\mathbf{A}^*$ are deviatoric. We can use (1.15) and (1.21) to express $D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}]$ in terms of β instead of θ:

$$D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}] = \beta(\mathbf{A})\,\mathbf{H} + \gamma(\mathbf{A})(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A})\,, \quad \forall\,\mathbf{H} \in \mathcal{P}(\mathbf{A})^{\perp}. \tag{3.9}$$

Similarly, from (2.22), (1.15), and (1.21), we have

$$D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}]\cdot\mathbf{H} = \mathrm{tr}(\mathbf{EH})\beta(\mathbf{A}) + 2\,\mathrm{tr}(\mathbf{AEH})\gamma(\mathbf{A}), \quad \forall\,\mathbf{H}\in\mathcal{P}(\mathbf{A})^\perp. \tag{3.10}$$

On taking the inner product of (3.8) or (3.9) with $\mathbf{E}$, we obtain the same expression as on the right-hand side of (2.22) or (3.10), respectively. Therefore

$$D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}]\cdot\mathbf{E} = D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}]\cdot\mathbf{H}, \quad \forall\,\mathbf{H}\in\mathcal{P}(\mathbf{A})^\perp. \tag{3.11}$$

Since $\mathbf{A}^*\mathbf{C}$ and $\mathbf{AC}\in\mathcal{P}(\mathbf{A})$ for any $\mathbf{C}\in\mathcal{P}(\mathbf{A})$, by setting $\mathbf{E}=\mathbf{C}$ in (2.22) or (3.10) and using (3.11), we obtain

$$D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}]\cdot\mathbf{C} = D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{C}]\cdot\mathbf{H} = 0, \quad \forall\,\mathbf{C}\in\mathcal{P}(\mathbf{A}),\ \forall\,\mathbf{H}\in\mathcal{P}(\mathbf{A})^\perp. \tag{3.12}$$

By setting $\mathbf{E}=\mathbf{H}_l$ and $\mathbf{H}=\mathbf{H}_m$ in (2.22) and using (3.2) and (3.6), we obtain

$$D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l]\cdot\mathbf{H}_m = 0\,. \tag{3.13}$$

By (3.7) we may set $\mathbf{H}=\sum_{k=1}^3\mathbf{H}_k$ in (2.28), and when (3.13) is applied to this result, we obtain

$$\theta(\mathbf{A}) = \frac{1}{6}\sum_{k=1}^{3} D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_k]\cdot\mathbf{H}_k\,. \tag{3.14}$$

In (3.14) *and* (3.12), *and in similar results below, the orthonormal basis* $\{\mathbf{e}_i\}$ *relative to which* $\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3$ *are defined is assumed to be a principal basis for* $\mathbf{A}$. Also, since only differentiablity of $\mathbf{\Phi}^*$ was assumed above, the relations (3.8)–(3.14) are generally valid only for $\mathbf{A}\in\mathcal{U}_3$.

Now assume that $\mathbf{\Phi}$ is differentiable. From (1.16) we have

$$D\mathbf{\Phi}(\mathbf{A})[\mathbf{E}] = D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}] + \tfrac{1}{3}[\nabla(\mathrm{tr}\,\mathbf{\Phi})(\mathbf{A})\cdot\mathbf{E}]\,\mathbf{I}. \tag{3.15}$$

Since $\mathbf{I}\cdot\mathbf{G}^*=0$ for any tensor $\mathbf{G}$, it follows that

$$D\mathbf{\Phi}(\mathbf{A})[\mathbf{E}]\cdot\mathbf{G}^* = D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}]\cdot\mathbf{G}^*\,. \tag{3.16}$$

From (3.15) and (2.23) with $\psi=\mathrm{tr}\,\mathbf{\Phi}$, it follows that

$$D\mathbf{\Phi}(\mathbf{A})[\mathbf{H}] = D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}], \quad \forall\,\mathbf{H}\in\mathcal{P}(\mathbf{A})^\perp\,. \tag{3.17}$$

These results hold for each $\mathbf{A}\in\mathcal{U}$. Now assume that $\mathbf{A}\in\mathcal{U}_3$. Then by (3.17) we may replace $\mathbf{\Phi}^*$ by $\mathbf{\Phi}$ in (3.8) and (3.9). Similarly, since $\mathbf{H}^*=\mathbf{H}$ for each $\mathbf{H}\in\mathcal{P}(\mathbf{A})^\perp$, by (3.16) and (3.17) we may replace $\mathbf{\Phi}^*$ by $\mathbf{\Phi}$ in (2.22), (2.28), (2.33), and (3.10)–(3.14). If a scalar-valued isotropic function ψ on $\mathcal{U}$ is twice-differentiable on $\mathcal{U}_3$, then by applying (3.12) and (3.13) with $\mathbf{\Phi}^*\to\mathbf{\Phi}=\nabla\psi$, we see that for any $\mathbf{A}\in\mathcal{U}_3$,

$$D^2\psi(\mathbf{A})[\mathbf{C},\mathbf{H}] = 0, \quad \forall\,\mathbf{C}\in\mathcal{P}(\mathbf{A}),\ \forall\,\mathbf{H}\in\mathcal{P}(\mathbf{A})^\perp, \tag{3.18}$$

and

$$D^2\psi(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_m] = 0\,, \quad \nabla\psi(\mathbf{A})\cdot\mathbf{H}_l = 0\,, \tag{3.19}$$

where the relation on the right is just a special case of (2.23).

Now assume that $\mathbf{\Phi}^*$ is C^1. Then $D\mathbf{\Phi}^*$ is continuous, and without loss of generality we may assume that θ is also continuous on $\mathcal{U}$ (cf. Theorem 2.1). On taking the limit of (3.13) or (3.14) as $\mathbf{A} \to \mathbf{B} = \sum_{i=1}^3 b_i\mathbf{e}_i \otimes \mathbf{e}_i \in \mathcal{U}_1 \cup \mathcal{U}_2$ from within $\mathcal{U}_3 \cap S\{\mathbf{e}_i\}$ while holding the $\mathbf{H}_k$ fixed, and then writing $\mathbf{A}$ in place of $\mathbf{B}$, we see that (3.13) and (3.14) hold for every $\mathbf{A} \in \mathcal{U}$. By applying the same limiting process to (3.11) and (3.12) and using (2.17), we see that for every $\mathbf{A} \in \mathcal{U}$ with principal basis $\{\mathbf{e}_i\}$,

$$D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}]\cdot\mathbf{C} = D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{C}]\cdot\mathbf{H} = 0\,, \quad \forall\,\mathbf{C} \in S\{\mathbf{e}_i\},\ \forall\,\mathbf{H} \in S\{\mathbf{e}_i\}^\perp, \tag{3.20}$$

and $D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}]\cdot\mathbf{E} = D\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}]\cdot\mathbf{H}$ for each $\mathbf{H} \in S\{\mathbf{e}_i\}^\perp$. In the latter relation $\mathbf{E}$ is arbitrary, so from (2.18) we conclude that (3.11) holds for each $\mathbf{A} \in \mathcal{U}$. From the previous paragraph it follows that these results also hold with $\mathbf{\Phi}^*$ replaced by $\mathbf{\Phi}$, provided that $\mathbf{\Phi}$ is C^1, or more generally, that $\mathbf{\Phi}^*$ is C^1 and $\operatorname{tr}\mathbf{\Phi}$ is differentiable, since the latter conditions suffice for (3.16) and (3.17).

Now assume that $\mathbf{\Phi}^*$ is twice differentiable, and let $\mathbf{A} \in \mathcal{U}_3$. By setting $\mathbf{E} = \mathbf{H}_l$ and $\mathbf{F} = \mathbf{H}_m$ in (2.10) and using (3.19) and $(3.3)_1$, we obtain

$$D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_m] = \gamma(\mathbf{A})\,\mathbf{H}_n\,. \tag{3.21}$$

On taking the inner product of (3.21) with $\mathbf{H}_n$ or $\mathbf{H}_m$ and using (3.2), or from (2.25) and (3.3), we obtain

$$\gamma(\mathbf{A}) = \tfrac{1}{2}D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_m]\cdot\mathbf{H}_n \tag{3.22}$$

and

$$D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_m]\cdot\mathbf{H}_m = 0\,. \tag{3.23}$$

Now assume that $\mathbf{\Phi}$ is twice differentiable. Then from (3.15) we have

$$D^2\mathbf{\Phi}(\mathbf{A})[\mathbf{E}, \mathbf{F}] = D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}, \mathbf{F}] + \tfrac{1}{3}D^2(\operatorname{tr}\mathbf{\Phi})(\mathbf{A})[\mathbf{E}, \mathbf{F}])\,\mathbf{I}\,, \tag{3.24}$$

which implies

$$D^2\mathbf{\Phi}(\mathbf{A})[\mathbf{E}, \mathbf{F}]\cdot\mathbf{G}^* = D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}, \mathbf{F}]\cdot\mathbf{G}^*\,. \tag{3.25}$$

These results hold for any $\mathbf{A} \in \mathcal{U}$. Now assume $\mathbf{A} \in \mathcal{U}_3$. Then from (3.25) it follows that we may replace $\mathbf{\Phi}^*$ by $\mathbf{\Phi}$ in the formulas (2.24), (2.25), (2.30), (3.22), and (3.23). By setting $\mathbf{E} = \mathbf{H}_l$ and $\mathbf{F} = \mathbf{H}_m$ in (3.24) and using (3.19) with $\psi = \operatorname{tr}\mathbf{\Phi}$, we obtain

$$D^2\mathbf{\Phi}(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_m] = D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_m]\,. \tag{3.26}$$

Thus (3.21) holds with $\mathbf{\Phi}^*$ replaced by $\mathbf{\Phi}$.

Now assume that $\mathbf{\Phi}^*$ is C^2. Then $D^2\mathbf{\Phi}^*$ is continuous, and without loss of generality we may assume that θ, γ, and β are also continuous on $\mathcal{U}$ (cf. Theorem 2.2).

Arguing as in the C^1 case above, we conclude that (3.8)–(3.10) and (3.21)–(3.23) hold for every $\mathbf{A} \in \mathcal{U}$. And from the previous paragraph it follows that these results hold with $\mathbf{\Phi}^*$ replaced by $\mathbf{\Phi}$ when $\mathbf{\Phi}$ is C^2. Actually, (3.8)–(3.10) with $\mathbf{\Phi}^* \to \mathbf{\Phi}$ hold for every $\mathbf{A} \in \mathcal{U}$ under the weaker condition that $\mathbf{\Phi}^*$ is C^2 and $\operatorname{tr}\mathbf{\Phi}$ is differentiable, since this condition suffices for (3.16) and (3.17). Similarly, (3.21)–(3.23) with $\mathbf{\Phi}^* \to \mathbf{\Phi}$ hold for every $\mathbf{A} \in \mathcal{U}$ under the weaker condition that $\mathbf{\Phi}^*$ is C^2 and $\operatorname{tr}\mathbf{\Phi}$ is twice differentiable.

4 Differentiability of the Coefficients

To establish sufficient conditions for the differentiablity of the coefficients we use the same approach as in the previous section, that is, for $\mathbf{A} \in \mathcal{U}_3$ we express $\nabla\theta(\mathbf{A})$ and $\nabla\gamma(\mathbf{A})$ in terms of $D^2\mathbf{\Phi}^*(\mathbf{A})$ or $D^3\mathbf{\Phi}^*(\mathbf{A})$ and show that these expressions have limits as $\mathbf{A} \to \mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$ from within $\mathcal{U}_3 \cap S\{\mathbf{e}_i\}$, where $\{\mathbf{e}_i\}$ is a principal basis for $\mathbf{B}$. Then to conclude that θ and γ are C^1 functions on $\mathcal{U}$, we need an appropriate generalization of Proposition 2.1. Since we are also interested in analogous results for higher order derivatives of the coefficients, we generalize Proposition 2.1 to C^r isotropic functions, where r denotes some positive integer.

Proposition 4.1 *Let ψ be a C^{r-1} scalar-valued isotropic function on $\mathcal{U}$ which is C^r on $\mathcal{U}_3$. Then the following conditions are equivalent:*

(1) ψ is C^r on $\mathcal{U}$;

(2) $\forall\, \mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$, the $\displaystyle\lim_{\substack{\mathbf{A}\to\mathbf{B} \\ \mathbf{A}\in\mathcal{U}_3}} D^r\psi(\mathbf{A})$ exists;

(3) $\forall\, \mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$, each principal basis $\{\mathbf{e}_i\}$ for $\mathbf{B}$, and any $\mathbf{E}_1, \ldots, \mathbf{E}_r \in Sym$, the $\displaystyle\lim_{\substack{\mathbf{A}\to\mathbf{B} \\ \mathbf{A}\in\mathcal{U}_3\cap S\{\mathbf{e}_i\}}} D^r\psi(\mathbf{A})[\mathbf{E}_1, \ldots, \mathbf{E}_r]$ exists.

Here C^0 (i.e., C^{r-1} for $r = 1$) means continuous. The proof is given in Section 6.[15] Throughout this section we assume that $\mathbf{\Phi}^*$ is at least C^2 on $\mathcal{U}$. Then without loss of generality we can assume that β, γ, and θ are continuous on $\mathcal{U}$ (cf. Theorem 2.2). As in the previous section, $\mathbf{E}, \mathbf{F}, \mathbf{G}$ denote arbitrary symmetric tensors. Unless stated otherwise, l, m, n denotes an arbitrary permutation of $1, 2, 3$.

By setting $\mathbf{K} = \mathbf{H} \in \mathcal{H}\{\mathbf{e}_i\}$ in (2.24) and using (2.27), we obtain the following formula for $\nabla\theta$ on $\mathcal{U}_3$:

$$\nabla\theta(\mathbf{A}) \cdot \mathbf{E} = \tfrac{1}{6} D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}, \mathbf{H}] \cdot \mathbf{H} - \tfrac{1}{3}\operatorname{tr}(\mathbf{E}^*\mathbf{H}^2)\gamma(\mathbf{A}), \quad \forall\, \mathbf{H} \in \mathcal{H}\{\mathbf{e}_i\}, \tag{4.1}$$

[15] As shown in Section 6, condition (3) can be replaced by a much weaker condition. However, the version stated above suffices for the applications in this paper.

where $\{\mathbf{e}_i\}$ is any principal basis for $\mathbf{A}$. Let $\mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$ and let $\{\mathbf{e}_i\}$ be any principal basis for $\mathbf{B}$. Choose any $\mathbf{H} \in \mathcal{H}\{\mathbf{e}_i\}$ and any symmetric $\mathbf{E}$. Then the formula for $\nabla\theta(\mathbf{A})$ in (4.1) holds for every $\mathbf{A} \in \mathcal{U}_3 \cap S\{\mathbf{e}_i\}$. Since $D\mathbf{\Phi}^*$ and γ are continuous on $\mathcal{U}$, the right-hand side of (4.1) has a limit as $\mathbf{A} \to \mathbf{B}$ from within $\mathcal{U}_3 \cap S\{\mathbf{e}_i\}$, and thus so does $D\theta(\mathbf{A})[\mathbf{E}] = \nabla\theta(\mathbf{A}) \cdot \mathbf{E}$. Since θ is isotropic and continuous on $\mathcal{U}$ and C^1 (in fact C^2) on $\mathcal{U}_3$, Proposition 4.1 with $\psi = \theta$ and $r = 1$ yields

Theorem 4.1 *If $\mathbf{\Phi}^*$ is C^2 on $\mathcal{U}$, then θ is C^1 on $\mathcal{U}$, and $\nabla\theta$ is given by (4.1) for any $\mathbf{A} \in \mathcal{U}$ and any principal basis $\{\mathbf{e}_i\}$ for $\mathbf{A}$.*

Now assume that $\mathbf{\Phi}^*$ is C^3. By differentiating (2.10) and using (2.9), we obtain the following formula for $D^3\mathbf{\Phi}^*(\mathbf{A})$ at any $\mathbf{A} \in \mathcal{U}_3$:

$$\begin{aligned} D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E},\mathbf{F},\mathbf{G}] \;=\;& D^3\theta(\mathbf{A})[\mathbf{E},\mathbf{F},\mathbf{G}]\,\mathbf{A}^* + D^3\gamma(\mathbf{A})[\mathbf{E},\mathbf{F},\mathbf{G}]\,\mathbf{A}^{**} \\ &+ D^2\theta(\mathbf{A})[\mathbf{E},\mathbf{F}]\,\mathbf{G}^* + D^2\gamma(\mathbf{A})[\mathbf{E},\mathbf{F}](\mathbf{A}^*\mathbf{G}^* + \mathbf{G}^*\mathbf{A}^*)^* \\ &+ D^2\theta(\mathbf{A})[\mathbf{E},\mathbf{G}]\,\mathbf{F}^* + D^2\gamma(\mathbf{A})[\mathbf{E},\mathbf{G}](\mathbf{A}^*\mathbf{F}^* + \mathbf{F}^*\mathbf{A}^*)^* \\ &+ D^2\theta(\mathbf{A})[\mathbf{F},\mathbf{G}]\,\mathbf{E}^* + D^2\gamma(\mathbf{A})[\mathbf{F},\mathbf{G}](\mathbf{A}^*\mathbf{E}^* + \mathbf{E}^*\mathbf{A}^*)^* \\ &+ (\nabla\gamma(\mathbf{A})\cdot\mathbf{E})(\mathbf{G}^*\mathbf{F}^* + \mathbf{F}^*\mathbf{G}^*)^* \\ &+ (\nabla\gamma(\mathbf{A})\cdot\mathbf{F})(\mathbf{G}^*\mathbf{E}^* + \mathbf{E}^*\mathbf{G}^*)^* \\ &+ (\nabla\gamma(\mathbf{A})\cdot\mathbf{G})(\mathbf{E}^*\mathbf{F}^* + \mathbf{F}^*\mathbf{E}^*)^* . \end{aligned} \tag{4.2}$$

On taking the inner product of (4.2) with $\mathbf{H} \in \mathcal{P}(\mathbf{A})^\perp$ and using (2.12), (2.19), and (2.20), we obtain

$$\begin{aligned} D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E},\mathbf{F},\mathbf{G}]\cdot\mathbf{H} \;=\;& D^2\theta(\mathbf{A})[\mathbf{E},\mathbf{F}]\,\mathrm{tr}(\mathbf{G}\mathbf{H}) + 2D^2\gamma(\mathbf{A})[\mathbf{E},\mathbf{F}]\,\mathrm{tr}(\mathbf{A}^*\mathbf{G}\mathbf{H}) \\ &+ D^2\theta(\mathbf{A})[\mathbf{E},\mathbf{G}]\,\mathrm{tr}(\mathbf{F}\mathbf{H}) + 2D^2\gamma(\mathbf{A})[\mathbf{E},\mathbf{G}]\,\mathrm{tr}(\mathbf{A}^*\mathbf{F}\mathbf{H}) \\ &+ D^2\theta(\mathbf{A})[\mathbf{F},\mathbf{G}]\,\mathrm{tr}(\mathbf{E}\mathbf{H}) + 2D^2\gamma(\mathbf{A})[\mathbf{F},\mathbf{G}]\,\mathrm{tr}(\mathbf{A}^*\mathbf{E}\mathbf{H}) \\ &+ 2(\nabla\gamma(\mathbf{A})\cdot\mathbf{G})\,\mathrm{tr}(\mathbf{E}^*\mathbf{F}^*\mathbf{H}) + 2(\nabla\gamma(\mathbf{A})\cdot\mathbf{F})\,\mathrm{tr}(\mathbf{G}^*\mathbf{E}^*\mathbf{H}) \\ &+ 2(\nabla\gamma(\mathbf{A})\cdot\mathbf{E})\,\mathrm{tr}(\mathbf{G}^*\mathbf{F}^*\mathbf{H}) , \end{aligned} \tag{4.3}$$

for any $\mathbf{A} \in \mathcal{U}_3$ and any $\mathbf{H} \in \mathcal{P}(\mathbf{A})^\perp$. We wish to solve this for $\nabla\gamma(\mathbf{A})\cdot\mathbf{E}$. Set $\mathbf{F} = \mathbf{H}_l$, $\mathbf{G} = \mathbf{H}_m$, and $\mathbf{H} = \mathbf{H}_n$. Then the first two lines in (4.3) drop out from (3.2) and (3.6); and the third and fourth lines drop out from (3.19). On using $(3.3)_2$ in the last line, we obtain the following simple formula for $\nabla\gamma$ on $\mathcal{U}_3$:

$$\nabla\gamma(\mathbf{A})\cdot\mathbf{E} = \tfrac{1}{2}D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E},\mathbf{H}_l,\mathbf{H}_m]\cdot\mathbf{H}_n , \tag{4.4}$$

where the orthonormal basis $\{\mathbf{e}_i\}$ relative to which $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ are defined (cf. (3.1)) is any principal basis for $\mathbf{A}$. Arguing as in the proof of Theorem 4.1, we conclude that γ is C^1 on $\mathcal{U}$ when $\mathbf{\Phi}^*$ is C^3, and that (4.4) holds for each $\mathbf{A} \in \mathcal{U}$.

Next we wish to show that θ is C^2 on $\mathcal{U}$ when $\mathbf{\Phi}^*$ is C^3. Since θ is C^1 on $\mathcal{U}$ (cf. Theorem 4.1), by Proposition 4.1 we need only show that $\lim_{\mathbf{A}\to\mathbf{B}} D^2\theta(\mathbf{A})[\mathbf{E},\mathbf{F}]$,

$\mathbf{A} \in \mathcal{U}_3 \cap \mathcal{S}\{\mathbf{e}_i\}$, exists for each $\mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$, each principal basis $\{\mathbf{e}_i\}$ for $\mathbf{B}$, and any symmetric tensors $\mathbf{E}, \mathbf{F}$. And since $\{\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3\}$ is a basis for $\mathcal{S}\{\mathbf{e}_i\}^\perp$, it suffices to show that $D^2\theta(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_m]$, $D^2\theta(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l]$, $D^2\theta(\mathbf{A})[\mathbf{C}, \mathbf{D}]$, and $D^2\theta(\mathbf{A})[\mathbf{C}, \mathbf{H}_l]$ have limits from within $\mathcal{U}_3 \cap \mathcal{S}\{\mathbf{e}_i\}$, for any $\mathbf{C}, \mathbf{D} \in \mathcal{S}\{\mathbf{e}_i\}$. But the first and last of these expressions are zero by (3.19) and (3.18), respectively. Consider the third expression. Set $\mathbf{E} = \mathbf{C}$, $\mathbf{F} = \mathbf{D}$, and $\mathbf{G} = \mathbf{K}$ in (4.3), where $\mathbf{C}, \mathbf{D} \in \mathcal{P}(\mathbf{A})$ and $\mathbf{K} \in \mathcal{P}(\mathbf{A})^\perp$. Then $\mathbf{A}^*\mathbf{C}, \mathbf{A}^*\mathbf{D}, \mathbf{C}^*\mathbf{D}^* \in \mathcal{P}(\mathbf{A})$, so the second and third lines and the first expression in the fourth line in (4.3) drop out since $\mathbf{H} \in \mathcal{P}(\mathbf{A})^\perp$. Thus for any $\mathbf{A} \in \mathcal{U}_3$ we have

$$\begin{aligned} D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{C}, \mathbf{D}, \mathbf{K}] \cdot \mathbf{H} \;\; = \;\; & D^2\theta(\mathbf{A})[\mathbf{C}, \mathbf{D}] \operatorname{tr}(\mathbf{KH}) + 2D^2\gamma(\mathbf{A})[\mathbf{C}, \mathbf{D}] \operatorname{tr}(\mathbf{A}^*\mathbf{KH}) \\ & + 2(\nabla\gamma(\mathbf{A}) \cdot \mathbf{D}) \operatorname{tr}(\mathbf{C}^*\mathbf{KH}) + 2(\nabla\gamma(\mathbf{A}) \cdot \mathbf{C}) \operatorname{tr}(\mathbf{D}^*\mathbf{KH}), \\ & \forall\, \mathbf{C}, \mathbf{D} \in \mathcal{P}(\mathbf{A}), \quad \forall\, \mathbf{H}, \mathbf{K} \in \mathcal{P}(\mathbf{A})^\perp. \end{aligned} \tag{4.5}$$

Now let $\mathbf{K} = \mathbf{H} \in \mathcal{H}\{\mathbf{e}_i\}$ in (4.5), where $\{\mathbf{e}_i\}$ is any principal basis for $\mathbf{A} \in \mathcal{U}_3$. Then $\mathcal{P}(\mathbf{A}) = \mathcal{S}\{\mathbf{e}_i\}$, so from (2.27) we obtain

$$D^2\theta(\mathbf{A})[\mathbf{C}, \mathbf{D}] = \tfrac{1}{6} D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{C}, \mathbf{D}, \mathbf{H}] \cdot \mathbf{H}, \quad \forall\, \mathbf{C}, \mathbf{D} \in \mathcal{S}\{\mathbf{e}_i\},\ \forall\, \mathbf{H} \in \mathcal{H}\{\mathbf{e}_i\}. \tag{4.6}$$

Since $\mathbf{\Phi}^*$ is C^3, it follows that $D^2\theta(\mathbf{A})[\mathbf{C}, \mathbf{D}]$ has a limit as $\mathbf{A} \to \mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$ from within $\mathcal{U}_3 \cap \mathcal{S}\{\mathbf{e}_i\}$, with $\mathbf{C}, \mathbf{D} \in \mathcal{S}\{\mathbf{e}_i\}$ fixed. Finally, consider the expression $D^2\theta(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l]$. On settting $\mathbf{E} = \mathbf{F} = \mathbf{H}_l$ and $\mathbf{G} = \mathbf{H} = \mathbf{H}_l$ or $\mathbf{H}_m$ in (4.3), and using (3.2), (3.5), and (3.19), we obtain

$$\tfrac{1}{6} D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l, \mathbf{H}_l] \cdot \mathbf{H}_l = D^2\theta(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l] - a_l^* D^2\gamma(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l] \tag{4.7}$$

and

$$\tfrac{1}{2} D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l, \mathbf{H}_m] \cdot \mathbf{H}_m = D^2\theta(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l] - a_m^* D^2\gamma(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l]. \tag{4.8}$$

On adding (4.7) and (4.8) and using $-a_l^* - a_m^* = a_n^*$, we obtain

$$\begin{aligned} 2D^2\theta(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l] + a_n^* D^2\gamma(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l] \;\; = \;\; & \tfrac{1}{6} D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l, \mathbf{H}_l] \cdot \mathbf{H}_l + \\ & \tfrac{1}{2} D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l, \mathbf{H}_m] \cdot \mathbf{H}_m. \end{aligned} \tag{4.9}$$

Then by interchanging m and n in (4.9) and adding the result to (4.8), we obtain

$$\begin{aligned} 3D^2\theta(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l] \;\; = \;\; & \tfrac{1}{2} D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l, \mathbf{H}_m] \cdot \mathbf{H}_m + \tfrac{1}{2} D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l, \mathbf{H}_n] \cdot \mathbf{H}_n \\ & + \tfrac{1}{6} D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l, \mathbf{H}_l] \cdot \mathbf{H}_l, \end{aligned} \tag{4.10}$$

where the orthonormal basis $\{\mathbf{e}_i\}$ relative to which $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ are defined is any principal basis for $\mathbf{A}$. It follows that $D^2\theta(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l]$ has a limit as $\mathbf{A} \to \mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$ from within $\mathcal{U}_3 \cap \mathcal{S}\{\mathbf{e}_i\}$ with $\mathbf{H}_l$ fixed. From the remarks at the beginning of the paragraph, we conclude that θ is C^2 on $\mathcal{U}$ when $\mathbf{\Phi}^*$ is C^3. On combining this with the result of the previous paragraph and using (1.21) and (1.18), we obtain

Theorem 4.2 *If $\mathbf{\Phi}^*$ is C^3 on $\mathcal{U}$, then θ is C^2 on $\mathcal{U}$, and γ and β are C^1 on $\mathcal{U}$. For any $\mathbf{A} \in \mathcal{U}$, $\nabla\theta(\mathbf{A})$ and $\nabla\gamma(\mathbf{A})$ are given by (4.1) and (4.4), respectively, and*

$$\nabla\beta(\mathbf{A}) \cdot \mathbf{E} = \nabla\theta(\mathbf{A}) \cdot \mathbf{E} - \tfrac{2}{3}(I_{\mathbf{E}}\gamma(\mathbf{A}) + I_{\mathbf{A}}\nabla\gamma(\mathbf{A}) \cdot \mathbf{E}). \tag{4.11}$$

If, in addition, $\operatorname{tr}\mathbf{\Phi}$ *is C^1 on $\mathcal{U}$, then so is α, and $\nabla\alpha$ can be expressed in terms of $\nabla(\operatorname{tr}\mathbf{\Phi})$, $\nabla\beta$, and $\nabla\gamma$ by means of the relation*

$$\begin{aligned}\nabla(\operatorname{tr}\mathbf{\Phi})(\mathbf{A}) \cdot \mathbf{E} &= (3\nabla\alpha(\mathbf{A}) + I_{\mathbf{A}}\nabla\beta(\mathbf{A}) + \mathit{II}_{\mathbf{A}}\nabla\gamma(\mathbf{A})) \cdot \mathbf{E} \\ &\quad + I_{\mathbf{E}}\beta(\mathbf{A}) + 2(\mathbf{A} \cdot \mathbf{E})\gamma(\mathbf{A}).\end{aligned} \tag{4.12}$$

Now we turn to the smoothness of γ at points in $\mathcal{U}_2$. Let $\mathbf{B} = \sum_{i=1}^{3} b_i \mathbf{e}_i \otimes \mathbf{e}_i$ be any spectral decomposition of $\mathbf{B} \in \mathcal{U}_2$. Then $b_l \neq b_m = b_n$ for some permutation l, m, n of $1, 2, 3$. For $\mathbf{H}, \mathbf{K} \in \mathcal{S}\{\mathbf{e}_i\}^\perp$ satisfying (2.31) we have, by (2.32) and (2.24),

$$2(a_l - a_m)\nabla\gamma(\mathbf{A}) \cdot \mathbf{E} = D^2\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}, \mathbf{K}] \cdot \mathbf{H} - 2\operatorname{tr}(\mathbf{E}^*\mathbf{K}\mathbf{H})\gamma(\mathbf{A}) \tag{4.13}$$

for any $\mathbf{A} \in \mathcal{U}_3$ with principal basis $\{\mathbf{e}_i\}$. Then from (4.13), Proposition 4.1, and arguments similar to those preceding Theorem 2.3, we conclude that γ is C^1 on $\mathcal{U}_2 \cup \mathcal{U}_3$ if $\mathbf{\Phi}^*$ is C^2 on $\mathcal{U}_2 \cup \mathcal{U}_3$. Now assume that $\mathbf{\Phi}^*$ is C^3 on $\mathcal{U}_2 \cup \mathcal{U}_3$. Then $\nabla\gamma$ is continuous on $\mathcal{U}_2 \cup \mathcal{U}_3$, and for $\mathbf{H}$ and $\mathbf{K}$ as above, (4.5) yields

$$\begin{aligned}D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{C}, \mathbf{D}, \mathbf{K}] \cdot \mathbf{H} &= 2(a_l - a_m)D^2\gamma(\mathbf{A})[\mathbf{C}, \mathbf{D}] + 2(c_l - c_m)(\nabla\gamma(\mathbf{A}) \cdot \mathbf{D}) \\ &\quad + 2(d_l - d_m)(\nabla\gamma(\mathbf{A}) \cdot \mathbf{C})\end{aligned} \tag{4.14}$$

for any $\mathbf{A} \in \mathcal{U}_3 \cap \mathcal{S}\{\mathbf{e}_i\}$ and any $\mathbf{C}, \mathbf{D} \in \mathcal{S}\{\mathbf{e}_i\}$. By subtracting (4.7) from (4.8), we obtain

$$\begin{aligned}(a_l - a_m)D^2\gamma(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l] &= \tfrac{1}{2}D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l, \mathbf{H}_m] \cdot \mathbf{H}_m \\ &\quad - \tfrac{1}{6}D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{H}_l, \mathbf{H}_l, \mathbf{H}_l] \cdot \mathbf{H}_l.\end{aligned} \tag{4.15}$$

Arguing as above (see also the comments preceding (4.5)), we conclude that γ is C^2 on $\mathcal{U}_2 \cup \mathcal{U}_3$. On combining the results of this paragraph with Theorems 4.1 and 4.2 and using (1.21) and (1.18), we obtain

Theorem 4.3 *If $\mathbf{\Phi}^*$ is C^2 (resp. C^3) on $\mathcal{U}_2 \cup \mathcal{U}_3$, then θ, γ, β are C^1 (resp. C^2) on $\mathcal{U}_2 \cup \mathcal{U}_3$. If, in addition,* $\operatorname{tr}\mathbf{\Phi}$ *is C^1 (resp. C^2) on $\mathcal{U}_2 \cup \mathcal{U}_3$, then so is α.*

Of course, if $\mathbf{\Phi}$ is C^k then $\mathbf{\Phi}^*$ and $\operatorname{tr}\mathbf{\Phi}$ are C^k, so Theorems 4.1, 4.2, and 4.3 yield Theorem 1.2 and the second part of Theorem 1.3. From (3.25) it follows that we may replace $\mathbf{\Phi}^*$ with $\mathbf{\Phi}$ in (4.1) and (4.13) when $\mathbf{\Phi}$ is C^2. If $\mathbf{\Phi}$ is C^3, then by differentiating (3.24) we obtain

$$D^3\mathbf{\Phi}(\mathbf{A})[\mathbf{E}, \mathbf{F}, \mathbf{G}] \cdot \mathbf{L}^* = D^3\mathbf{\Phi}^*(\mathbf{A})[\mathbf{E}, \mathbf{F}, \mathbf{G}] \cdot \mathbf{L}^* \tag{4.16}$$

for any symmetric tensors $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{L}$, so we may replace $\mathbf{\Phi}^*$ by $\mathbf{\Phi}$ in (4.3)–(4.10), (4.14), and (4.15).

5 Smoothness of the Representations for the Coefficients

If ψ is a scalar-valued isotropic function on $\mathcal{U}$, then there are scalar-valued functions $\hat{\psi}$, $\bar{\psi}$, $\tilde{\psi}$, and ψ_* on appropriate subsets of $\mathbb{R}^3$, such that

$$\begin{aligned}\psi(\mathbf{A}) &= \hat{\psi}(I_\mathbf{A}, II_\mathbf{A}, III_\mathbf{A}) = \bar{\psi}(I_\mathbf{A}, \overline{II}_\mathbf{A}, \overline{III}_\mathbf{A}) \\ &= \tilde{\psi}(a_1, a_2, a_3) = \psi_*(I_\mathbf{A}, \overline{II}_{\mathbf{A}^*}, \overline{III}_{\mathbf{A}^*}),\end{aligned} \tag{5.1}$$

with $\tilde{\psi}$ symmetric in its arguments.[16] Conversely, if $\psi : \mathcal{U} \to \mathbb{R}$ has any one of the representations above, then ψ is isotropic. In the statements and proofs of our theorems on the smoothness of the coefficients α, β, γ, and θ, we have regarded these coefficients as functions on $\mathcal{U}$. Of course, since the coefficient functions are isotropic, they each have representations of the form (5.1). We are then faced with the question of whether the established smoothness of the coefficients, regarded as isotropic scalar-valued functions on $\mathcal{U}$, is inherited by the corresponding functions in the representations. The answer depends on the type of smoothness considered and on which representation is used. The reader is referred to Ball [13] for a thorough discussion of the relationship between the smoothness of ψ, $\hat{\psi}$, and $\tilde{\psi}$. We will discuss some of his results below.

The easiest case is when "smooth" is interpreted as continuous. For this case we have the following simple result. *Continuity of any one of the five functions in (5.1) implies continuity of the others.* To see this, note that the arguments of any one of the functions $\hat{\psi}, \bar{\psi}, \psi_*$ can be expressed as polynomials in the arguments of the other two functions, so continuity of one of these functions implies continuity of the other two. Similarly, since the principal invariants, for example, can be expressed as symmetric polynomials in a_1, a_2, a_3, continuity of $\hat{\psi}$ implies continuity of $\tilde{\psi}$. Conversely, since the unordered triple (a_1, a_2, a_3) of eigenvalues of $\mathbf{A}$ is a continuous function of the principal invariants of $\mathbf{A}$,[17] continuity of $\tilde{\psi}$ implies continuity of $\hat{\psi}$. Since the principal invariants and the moments of $\mathbf{A}$ are continuous functions of $\mathbf{A}$, continuity of any one of the functions $\hat{\psi}, \bar{\psi}, \psi_*$ implies continuity of ψ. Finally, choose any orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathcal{V}$. Then

$$\tilde{\psi}(a_1, a_2, a_3) = \psi(a_1\mathbf{e}_1 \otimes \mathbf{e}_1 + a_2\mathbf{e}_2 \otimes \mathbf{e}_2 + a_3\mathbf{e}_3 \otimes \mathbf{e}_3), \tag{5.2}$$

so continuity of ψ implies continuity of $\tilde{\psi}$. *Therefore, Theorems 1.1, 1.3, and 2.1–2.3*

[16] Cf. Truesdell & Noll [4, §10] for a proof of the representations $\hat{\psi}, \bar{\psi}, \tilde{\psi}$. The representation ψ_* follows from $\bar{\psi}$ and the identities (2.2) and $\overline{III}_\mathbf{A} = \frac{1}{9}I_\mathbf{A}^3 + I_\mathbf{A}\overline{II}_{\mathbf{A}^*} + \overline{III}_{\mathbf{A}^*}$.

[17] As noted by Serrin [2, p. 463] and Ball [13, p. 709], this follows from the well-known fact that the roots of a polynomial are continuous functions of the coefficients, the polynomial in this case being the characteristic polynomial of $\mathbf{A}$. The roots are analytic functions of the coefficients on any domain in which the roots remain distinct.

also yield sufficient conditions for the continuity of the functions in the representations of the coefficients α, β, γ, and θ.[18]

The relationship between the smoothness of a scalar-valued isotropic function ψ on $\mathcal{U}$ and the smoothness of the corresponding functions in the representations (5.1) is more complicated when "smooth" denotes some degree of differentiability. We consider the simplest case first, namely, when $\psi(\mathbf{A})$ is expressed as a symmetric function of the eigenvalues of $\mathbf{A}$: $\psi(\mathbf{A}) = \tilde{\psi}(a_1, a_2, a_3)$. The domain $\tilde{\mathcal{U}}$ of $\tilde{\psi}$ is the set of all $(a_1, a_2, a_3) \in \mathbb{R}^3$ such that a_1, a_2, a_3 are the eigenvalues of some $\mathbf{A} \in \mathcal{U}$. If $(a_1, a_2, a_3) \in \tilde{\mathcal{U}}$, then $(a_l, a_m, a_n) \in \tilde{\mathcal{U}}$ for every permutation l, m, n of $1, 2, 3$, and $\mathbf{A} = \sum_{i=1}^3 a_i \mathbf{e}_i \otimes \mathbf{e}_i \in \mathcal{U}$ for any orthonormal basis $\{\mathbf{e}_i\}$ for $\mathcal{V}$. Then $\mathbf{A}_\delta = \sum_{i=1}^3 (a_i + \delta_i)\mathbf{e}_i \otimes \mathbf{e}_i \in \mathcal{U}$ for sufficiently small δ_i, since $\mathcal{U}$ is open and $\|\mathbf{A} - \mathbf{A}_\delta\| = (\sum_{i=1}^3 \delta_i^2)^{1/2}$. Hence $(a_1 + \delta_1, a_2 + \delta_2, a_3 + \delta_3) \in \tilde{\mathcal{U}}$ for sufficiently small δ_i, which shows that $\tilde{\mathcal{U}}$ is an open subset of $\mathbb{R}^3$. Since (5.2) holds for any fixed orthonormal basis $\{\mathbf{e}_i\}$ for $\mathcal{V}$, it follows that *smoothness of ψ implies smoothness of $\tilde{\psi}$* for any reasonable interpretation of the term "smooth". A looser but more descriptive statement of this result is that if $\psi(\mathbf{A})$ is a smooth function of $\mathbf{A}$, then ψ is a smooth function of the eigenvalues of $\mathbf{A}$.[19] *Therefore, Theorems 1.2, 1.3, and 4.1–4.3 also yield sufficient conditions for the coefficients $\alpha(\mathbf{A}), \beta(\mathbf{A}), \gamma(\mathbf{A})$, and $\theta(\mathbf{A})$ to be regarded as C^r ($r = 1, 2$) functions of the eigenvalues of* $\mathbf{A}$. In the other direction, Ball [13, Theorem 5.5] proved that *if $\tilde{\psi}$ is C^r then ψ is C^r, for $r = 0, 1, 2$.* He conjectured but could not prove that this holds for any positive integer r. However, his Theorem 5.7 does imply the weaker result that *if $\tilde{\psi}$ is C^{r+1} then ψ is C^r, for any positive integer r; hence ψ is C^∞ if $\tilde{\psi}$ is C^∞.*

The relationship between the smoothness of ψ and $\hat{\psi}$ is dramatically different. The domain of $\hat{\psi}$ is the set

$$\hat{\mathcal{U}} = \{(I_\mathbf{A}, II_\mathbf{A}, III_\mathbf{A}) : \mathbf{A} \in \mathcal{U}\} \subset \mathbb{R}^3 . \tag{5.3}$$

Then $\hat{\mathcal{U}} = \hat{\mathcal{U}}_1 \cup \hat{\mathcal{U}}_2 \cup \hat{\mathcal{U}}_3$, where

$$\hat{\mathcal{U}}_n = \{(I_\mathbf{A}, II_\mathbf{A}, III_\mathbf{A}) : \mathbf{A} \in \mathcal{U}_n\} \subset \mathbb{R}^3 . \tag{5.4}$$

Similarly, let $\tilde{\mathcal{U}}_n$ denote the set of all points $(a_1, a_2, a_3) \in \tilde{\mathcal{U}}$ such that n of the a_i are distinct; equivalently, $(a_1, a_2, a_3) \in \tilde{\mathcal{U}}_n$ iff there is an $\mathbf{A} \in \mathcal{U}_n$ with eigenvalues a_1, a_2, a_3. Clearly, $\tilde{\mathcal{U}}_3$ is an open subset of $\mathbb{R}^3$. It can be shown that $\hat{\mathcal{U}}_3$ is also an open subset of $\mathbb{R}^3$; indeed, $\hat{\mathcal{U}}_3$ is the interior of $\hat{\mathcal{U}}$, whereas $\hat{\mathcal{U}}_1 \cup \hat{\mathcal{U}}_2$ is a subset of the

[18] Serrin [2] and Man [10] proceeded in the other direction. In their proofs of the continuity of the coefficients α, β, γ, they regarded these coefficients as symmetric functions of the eigenvalues.

[19] It is interesting to note (as did Ball [13, p. 701] in the same context) that the eigenvalues themselves (either regarded individually when arranged in ascending order or regarded as an unordered triple) fail to be differentiable at any $\mathbf{A} \in Sym$ for which $^\#\mathbf{A} < 3$, i.e., at any point in $Sym_1 \cup Sym_2$; cf. Kato [15, §II.6.4]. Thus, if ψ is smooth then ψ is the composition of a smooth function $\tilde{\psi} : \tilde{\mathcal{U}} \subset \mathbb{R}^3 \to \mathbb{R}$ with the *nondifferentiable* eigenvalue function $\mathbf{A} \mapsto (a_1, a_2, a_3)$.

boundary of $\hat{\mathcal{U}}$.[20] Thus, unlike $\mathcal{U}$ and $\tilde{\mathcal{U}}$, the domain $\hat{\mathcal{U}}$ of $\hat{\psi}$ is not an open set (unless $\mathcal{U} = \mathcal{U}_3$). Since the eigenvalues of $\mathbf{A}$ are analytic functions of the principal invariants of $\mathbf{A}$ on the open set $\hat{\mathcal{U}}_3$, the same arguments used for the case of continuity imply that *ψ is smooth on $\mathcal{U}_3$ iff $\hat{\psi}$ is smooth on $\hat{\mathcal{U}}_3$ iff $\tilde{\psi}$ is smooth on $\tilde{\mathcal{U}}_3$*. However, there can be a loss in the smoothness of $\hat{\psi}$ at points in $\hat{\mathcal{U}}_1 \cup \hat{\mathcal{U}}_2$. One complication which must be addressed in discussing the smoothness of $\hat{\psi}$ is the meaning to be assigned to the term "smooth" at these boundary points. For simplicity we now consider the case when $\mathcal{U} = Sym$; then $\hat{\mathcal{U}}_1 \cup \hat{\mathcal{U}}_2$ is the entire boundary of $\hat{\mathcal{U}}$. *If ψ is C^{3r} on $\mathcal{U} = Sym$ (and hence $\tilde{\psi}$ is C^{3r} on $\tilde{\mathcal{U}} = \mathbb{R}^3$), then $\hat{\psi}$ is C^r on $\hat{\mathcal{U}}$ in the sense that there is a C^r function f on $\mathbb{R}^3$ whose restriction to $\hat{\mathcal{U}}$ equals $\hat{\psi}$.*[21] In particular, if ψ is C^3 on $\mathcal{U} = Sym$, then $\hat{\psi}$ is C^1 on $\hat{\mathcal{U}}$, i.e., $\psi(\mathbf{A})$ may be regarded as a continuously differentiable function of the principal invariants of $\mathbf{A}$. However, the requirement that ψ be C^3 cannot be weakened to C^2. For suppose that ψ has the form

$$\psi(\mathbf{A}) = \tilde{\psi}(a_1, a_2, a_3) = g(a_1) + g(a_2) + g(a_3)\,, \tag{5.5}$$

where $g : \mathbb{R} \to \mathbb{R}$ is C^2 but not C^3. Then $\tilde{\psi}$ is C^2, and hence, by the theorem of Ball discussed in the previous paragraph, ψ is C^2. But by Theorem 3.8 in Ball [13], the fact that g is not C^3 implies that for ψ of the form (5.5), the derivative of $\hat{\psi}$ cannot be extended continuously from the interior of $\hat{\mathcal{U}}$ to the boundary of $\hat{\mathcal{U}}$, and hence $\hat{\psi}$ fails to be continuously differentiable (in any reasonable sense) at the boundary of $\hat{\mathcal{U}}$. In other words, *if $\psi(\mathbf{A})$ is only a C^2 function of $\mathbf{A}$, then ψ may fail to be a continuously differentiable function of the principal invariants of $\mathbf{A}$ at values of $I_{\mathbf{A}}, II_{\mathbf{A}}, III_{\mathbf{A}}$ for which $\Delta_{\mathbf{A}} = 0$, i.e., for which $\#\mathbf{A} < 3$.*

Note that none of the theorems in the previous sections yield sufficient conditions for the coefficients α, β, γ, or θ to be C^3. *Thus, none of our smoothness theorems guarantee that α, β, γ, or θ can be regarded as continuously differentiable functions of the principal invariants of* $\mathbf{A}$. Of course, since Proposition 4.1 is valid for arbitrary r, it might be possible to apply the techniques of Section 4 to obtain sufficient conditions for the continuity of higher order derivatives of the coefficients. For $r = 1, 2, 3$, we have shown that if $\mathbf{\Phi}$ is C^r on $\mathcal{U}$, then θ is C^{r-1} on $\mathcal{U}$, and α, β, γ are C^{r-1} on $\mathcal{U}_2 \cup \mathcal{U}_3$ and C^{r-2} on $\mathcal{U}$. It is reasonable to conjecture that this result also holds for $r > 3$. However, a glance at the formula (4.2) for $D^3\mathbf{\Phi}^*$ reveals that the formula for $D^4\mathbf{\Phi}^*$ will be quite messy, and it is not clear that the latter formula can be solved for $D^3\theta$ or $D^2\gamma$. Thus, even for the next simplest case, $r = 4$, the approach developed here might not be useful.

[20] Cf. Ball [13, Lemma 3.2] for this and other relations between the topological properties of $\hat{\mathcal{U}}$ and $\tilde{\mathcal{U}}$. That $\hat{\mathcal{U}}_1 \cup \hat{\mathcal{U}}_2$ lies in the boundary of $\hat{\mathcal{U}}$ also follows from the fact that $I_{\mathbf{A}}, II_{\mathbf{A}}, III_{\mathbf{A}}$ must satisfy the inequality $\Delta_{\mathbf{A}} \geq 0$ (cf. (1.13)), with $\Delta_{\mathbf{A}} = 0$ iff $\mathbf{A} \in \mathcal{U}_1 \cup \mathcal{U}_2$.

[21] Ball [13] attributes this result to Barbancon (1972). The result also follows from Theorem 3.2 in Ball's paper, which applies to more general (but not arbitrary) open invariant subsets $\mathcal{U}$ of Sym; cf. the discussion on pp. 700 and 705–706.

6 Proof of Proposition 4.1

Our proof of Proposition 4.1 employs Propositions 6.1 and 6.2 below. We begin with some definitions and results which are useful for a concise statement and proof of Proposition 6.1. Let *Orth* denote the set of orthogonal tensors. The *orbit* of a symmetric tensor $\mathbf{A}$ is $\{\mathbf{QAQ}^{\mathsf{T}} : \mathbf{Q} \in \mathit{Orth}\}$. Thus the orbit of $\mathbf{A}$ is the smallest invariant subset of *Sym* which contains $\mathbf{A}$; and $\mathcal{U} \subset \mathit{Sym}$ is invariant iff $\mathcal{U}$ contains the orbit of every $\mathbf{A} \in \mathcal{U}$. By an orbit of the invariant set $\mathcal{U}$ we mean a subset of $\mathcal{U}$ which is the orbit of some $\mathbf{A} \in \mathcal{U}$. Then $\mathcal{U}$ is the union of its orbits, and each orbit of $\mathcal{U}$ lies in one the the invariant subsets $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$. A scalar-valued or tensor-valued function on $\mathcal{U}$ is isotropic iff its restriction to each orbit is isotropic. In particular, a scalar-valued function on $\mathcal{U}$ is isotropic iff it is constant on every orbit of $\mathcal{U}$.

Consider a scalar-valued isotropic function ψ on $\mathcal{U}$. Let $\mathcal{T}_r(\mathit{Sym})$ denote the space of (covariant) tensors of order r on *Sym*, i.e., the space of all multilinear maps from $\mathit{Sym} \times \cdots \times \mathit{Sym}$ (r times) into $\mathbb{R}$. If ψ is r-times differentiable at the point $\mathbf{A} \in \mathcal{U}$, then $D^r\psi(\mathbf{A}) \in \mathcal{T}_r(\mathit{Sym})$. Let $\check{\mathcal{U}}$ denote the set of all points in $\mathcal{U}$ at which ψ is r-times differentiable, and let $\mathbf{\Psi}_r = D^r\psi : \check{\mathcal{U}} \to \mathcal{T}_r(\mathit{Sym})$. Then it is not hard to show that $\check{\mathcal{U}}$ is an invariant subset of $\mathcal{U}$ and

$$\mathbf{\Psi}_r(\mathbf{QAQ}^{\mathsf{T}})[\mathbf{QE}_1\mathbf{Q}^{\mathsf{T}}, \ldots, \mathbf{QE}_r\mathbf{Q}^{\mathsf{T}}] = \mathbf{\Psi}_r(\mathbf{A})[\mathbf{E}_1, \ldots, \mathbf{E}_r] \tag{6.1}$$

for each $\mathbf{A} \in \check{\mathcal{U}}$ and each $\mathbf{Q} \in \mathit{Orth}$; here and below, $\mathbf{E}_1, \ldots, \mathbf{E}_r$ denote arbitrary symmetric tensors.[22]

Now consider any invariant subset $\check{\mathcal{U}}$ of *Sym* and an arbitrary function $\mathbf{\Psi}_r : \check{\mathcal{U}} \to \mathcal{T}_r(\mathit{Sym})$. We say that $\mathbf{\Psi}_r$ is *isotropic* on $\check{\mathcal{U}}$ if (6.1) holds. This is consistent with our previous use of the term for functions from $\check{\mathcal{U}}$ into *Sym*. To see this, note that any $\mathbf{B} \in \mathcal{T}_1(\mathit{Sym})$ (the space of all linear functions from *Sym* into $\mathbb{R}$) can be identified with a unique $\mathbf{B} \in \mathit{Sym}$ via the canonical isomorphism $\mathbf{B}[\mathbf{E}] = \mathbf{B} \cdot \mathbf{E}$. Thus any function $\mathbf{\Psi}_1 : \check{\mathcal{U}} \to \mathit{Sym}$ may be identified with a function $\mathbf{\Psi}_1 : \check{\mathcal{U}} \to \mathcal{T}_1(\mathit{Sym})$. And if the latter function is isotropic in the sense of (6.1), we have

$$\begin{aligned} \mathbf{Q}^{\mathsf{T}}\mathbf{\Psi}_1(\mathbf{QAQ}^{\mathsf{T}})\mathbf{Q} \cdot \mathbf{E} &= \mathbf{\Psi}_1(\mathbf{QAQ}^{\mathsf{T}}) \cdot \mathbf{QEQ}^{\mathsf{T}} = \mathbf{\Psi}_1(\mathbf{QAQ}^{\mathsf{T}})[\mathbf{QEQ}^{\mathsf{T}}] \\ &= \mathbf{\Psi}_1(\mathbf{A})[\mathbf{E}] = \mathbf{\Psi}_1(\mathbf{A}) \cdot \mathbf{E}. \end{aligned}$$

Since this holds for every symmetric tensor $\mathbf{E}$, it follows that $\mathbf{Q}^{\mathsf{T}}\mathbf{\Psi}_1(\mathbf{QAQ}^{\mathsf{T}})\mathbf{Q} = \mathbf{\Psi}_1(\mathbf{A})$ and therefore $\mathbf{\Psi}_1(\mathbf{QAQ}^{\mathsf{T}}) = \mathbf{Q}\mathbf{\Psi}_1(\mathbf{A})\mathbf{Q}^{\mathsf{T}}$, i.e., $\mathbf{\Psi}_1$ is an isotropic tensor-valued function as defined in the Introduction.[23] Similarly, if we set $\mathcal{T}_0(\mathit{Sym}) = \mathbb{R}$ and

[22] Also, if $\mathbf{\Psi}_r = D^r\psi$ then $\mathbf{\Psi}_r(\mathbf{A})$ is a symmetric tensor of order r on *Sym*, i.e., $\mathbf{\Psi}_r[\mathbf{E}_1, \ldots, \mathbf{E}_r] = \mathbf{\Psi}_r[\mathbf{E}_{\sigma(1)}, \ldots, \mathbf{E}_{\sigma(r)}]$ for any permutation σ of $1, \ldots, r$; cf. (8.12.4) in Dieudonne [9]. However, this property of $\mathbf{\Psi}_r$ is not used for any of the results in this section.

[23] A similar argument shows that if $\check{\mathcal{U}}$ is the set of all points where the isotropic tensor-valued function $\mathbf{\Phi} : \mathcal{U} \to \mathit{Sym}$ is $r-1$ times differentiable, then $D^{r-1}\mathbf{\Phi}$ may be identified with an isotropic function $\mathbf{\Psi}_r : \check{\mathcal{U}} \to \mathcal{T}_r(\mathit{Sym})$.

interpret (6.1) for $r = 0$ as the condition $\mathbf{\Psi}_0(\mathbf{QAQ}^\top) = \mathbf{\Psi}_0(\mathbf{A})$, then $\mathbf{\Psi}_0$ is a scalar-valued isotropic function.

From (6.1) it follows that $\mathbf{\Psi}_r : \check{\mathcal{U}} \to \mathcal{T}_r(Sym)$ is isotropic iff its restriction to each orbit of $\check{\mathcal{U}}$ is isotropic. By setting $\mathbf{Q} = \mathbf{R}^\top$ and $\mathbf{A} = \mathbf{RBR}^\top$ in (6.1), we see that isotropy of $\mathbf{\Psi}_r$ implies

$$\mathbf{\Psi}_r(\mathbf{RBR}^\top)[\mathbf{E}_1, \ldots, \mathbf{E}_r] = \mathbf{\Psi}_r(\mathbf{B})[\mathbf{R}^\top\mathbf{E}_1\mathbf{R}, \ldots, \mathbf{R}^\top\mathbf{E}_r\mathbf{R}] \tag{6.2}$$

for any $\mathbf{B} \in \check{\mathcal{U}}$ and any $\mathbf{R} \in Orth$. Conversely, if for every orbit of $\check{\mathcal{U}}$ there is at least one tensor $\mathbf{B}$ in the orbit for which (6.2) holds for each $\mathbf{R} \in Orth$, then $\mathbf{\Psi}_r$ is isotropic. For if $\mathbf{A}$ is any tensor in the orbit of $\mathbf{B}$ then $\mathbf{A} = \bar{\mathbf{Q}}\mathbf{B}\bar{\mathbf{Q}}^\top$ for some $\bar{\mathbf{Q}} \in Orth$, so for any $\mathbf{Q} \in Orth$ we have

$$\begin{aligned}
\mathbf{\Psi}_r(\mathbf{QAQ}^\top)[\ldots, \mathbf{QE}_k\mathbf{Q}^\top, \ldots] &= \mathbf{\Psi}_r((\mathbf{Q}\bar{\mathbf{Q}})\mathbf{B}(\mathbf{Q}\bar{\mathbf{Q}})^\top)[\ldots, \mathbf{QE}_k\mathbf{Q}^\top, \ldots] \\
&= \mathbf{\Psi}_r(\mathbf{B})[\ldots, (\mathbf{Q}\bar{\mathbf{Q}})^\top(\mathbf{QE}_k\mathbf{Q}^\top)(\mathbf{Q}\bar{\mathbf{Q}}), \ldots] \\
&= \mathbf{\Psi}_r(\mathbf{B})[\ldots, \bar{\mathbf{Q}}^\top\mathbf{E}_k\bar{\mathbf{Q}}, \ldots] \\
&= \mathbf{\Psi}_r(\bar{\mathbf{Q}}\mathbf{B}\bar{\mathbf{Q}}^\top)[\ldots, \mathbf{E}_k, \ldots] \\
&= \mathbf{\Psi}_r(\mathbf{A})[\ldots, \mathbf{E}_k, \ldots]\,,
\end{aligned}$$

where the second equality follows from (6.2) with $\mathbf{R} = \mathbf{Q}\bar{\mathbf{Q}}$ and $\mathbf{E}_k \to \mathbf{QE}_k\mathbf{Q}^\top$, and the fourth equality follows from (6.2) with $\mathbf{R} = \bar{\mathbf{Q}}$. Given the value of $\mathbf{\Psi}_r$ at some point $\mathbf{B}$, we can use (6.2) to extend $\mathbf{\Psi}_r$ to an isotropic function on the orbit of $\mathbf{B}$.

Recall that the set $\mathcal{U}_3^<\{\mathbf{e}_i\} \subset \mathcal{U}_3 \cap S\{\mathbf{e}_i\}$ consists of all $\mathbf{A} = \sum_{i=1}^3 a_i\mathbf{e}_i \otimes \mathbf{e}_i \in \mathcal{U}$ for which $a_1 < a_2 < a_3$.

Proposition 6.1 *If $\mathbf{\Psi}_r : \mathcal{U}_3 \to \mathcal{T}_r(Sym)$ is continuous and isotropic on $\mathcal{U}_3$, then the following conditions are equivalent:*

(1) $\mathbf{\Psi}_r$ has a continuous extension $\hat{\mathbf{\Psi}}_r$ to $\mathcal{U}$;

(2) $\forall \mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$, the $\lim\limits_{\substack{\mathbf{A}\to\mathbf{B} \\ \mathbf{A}\in\mathcal{U}_3}} \mathbf{\Psi}_r(\mathbf{A})$ exists;

(3) $\forall \mathbf{B} \in \mathcal{U}_1 \cup \mathcal{U}_2$, each principal basis $\{\mathbf{e}_i\}$ for $\mathbf{B}$, and any $\mathbf{E}_1, \ldots, \mathbf{E}_r \in Sym$, the $\lim\limits_{\substack{\mathbf{A}\to\mathbf{B} \\ \mathbf{A}\in\mathcal{U}_3\cap S\{\mathbf{e}_i\}}} \mathbf{\Psi}_r(\mathbf{A})[\mathbf{E}_1, \ldots, \mathbf{E}_r]$ exists.

(4) for each orbit of $\mathcal{U}_1 \cup \mathcal{U}_2$ there is some $\mathbf{B}$ in the orbit and some principal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbf{B}$ corresponding to the ordered eigenvalues $b_1 \le b_2 \le b_3$ of $\mathbf{B}$, such that the $\lim\limits_{\substack{\mathbf{A}\to\mathbf{B} \\ \mathbf{A}\in\mathcal{U}_3^<\{\mathbf{e}_i\}}} \mathbf{\Psi}_r(\mathbf{A})[\mathbf{E}_1, \ldots, \mathbf{E}_r]$ exists for any $\mathbf{E}_1, \ldots, \mathbf{E}_r \in Sym$.

When these conditions hold, $\hat{\mathbf{\Psi}}_r$ is unique and isotropic on $\mathcal{U}$.

Proof: As in the proof of Proposition 2.1, the equivalence of (1) and (2), and the uniqueness $\hat{\boldsymbol{\Psi}}_r$, follows from the fact that $\mathcal{U}_3$ is dense in $\mathcal{U}$; and clearly (2) $\Rightarrow$ (3) $\Rightarrow$ (4). So assume (4) holds. By the linearity of $\boldsymbol{\Psi}_r$, (4) implies that $\lim_{\mathbf{A}\to\mathbf{B}} \boldsymbol{\Psi}_r(\mathbf{A})$, $\mathbf{A} \in \mathcal{U}_3^<\{\mathbf{e}_i\}$, exists. Let $\hat{\boldsymbol{\Psi}}_r(\mathbf{B}) \in \mathcal{T}_r(Sym)$ denote the value of this limit. Extend $\hat{\boldsymbol{\Psi}}_r$ to an isotropic function on the orbit of $\mathbf{B}$ by (6.2) with $\hat{\boldsymbol{\Psi}}_r$ in place of $\boldsymbol{\Psi}_r$. If $\hat{\mathbf{B}}$ lies in the orbit of $\mathbf{B}$ then $\hat{\mathbf{B}} = \sum_{i=1}^3 b_i \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i$ for some orthonormal basis $\{\hat{\mathbf{e}}_i\}$. Let $\{\mathbf{A}_n\}$ ($n = 1, 2, \ldots$) be any sequence in $\mathcal{U}_3$ converging to $\hat{\mathbf{B}}$. Let $\sum_{i=1}^3 a_{i,n}\mathbf{e}_{i,n}\otimes\mathbf{e}_{i,n}$, with $a_{1,n} < a_{2,n} < a_{3,n}$, be a spectral decomposition of $\mathbf{A}_n$; then $a_{i,n} \to b_i$ ($i = 1, 2, 3$) by the continuity of the ordered eigenvalues. Let $\mathbf{Q}_n$ be the orthogonal tensor which maps $\mathbf{e}_{i,n}$ to $\mathbf{e}_i$ ($i = 1, 2, 3$). Then $\mathbf{Q}_n\mathbf{A}_n\mathbf{Q}_n^{\mathsf{T}} = \sum_{i=1}^3 a_{i,n}\mathbf{e}_i \otimes \mathbf{e}_i \in \mathcal{U}_3^<\{\mathbf{e}_i\}$, and since $a_{i,n} \to b_i$ it follows that $\mathbf{Q}_n\mathbf{A}_n\mathbf{Q}_n^{\mathsf{T}} \to \mathbf{B}$. Thus (4) implies that $\boldsymbol{\Psi}_r(\mathbf{Q}_n\mathbf{A}_n\mathbf{Q}_n^{\mathsf{T}})[\mathbf{E}_1, \ldots, \mathbf{E}_r] \to \hat{\boldsymbol{\Psi}}_r(\mathbf{B})[\mathbf{E}_1, \ldots, \mathbf{E}_r]$ for any $\mathbf{E}_1, \ldots, \mathbf{E}_r$, and hence that $\boldsymbol{\Psi}_r(\mathbf{Q}_n\mathbf{A}_n\mathbf{Q}_n^{\mathsf{T}}) \to \hat{\boldsymbol{\Psi}}_r(\mathbf{B})$. Now the sequence $\{\mathbf{Q}_n\}$ need not converge, but since *Orth* is compact there is a subsequence $\{\mathbf{Q}_{n_k}\}$ such that $\mathbf{Q}_{n_k} \to \mathbf{Q}_\infty \in Orth$. Then $\mathbf{Q}_{n_k}\mathbf{A}_{n_k}\mathbf{Q}_{n_k}^{\mathsf{T}} \to \mathbf{Q}_\infty\hat{\mathbf{B}}\mathbf{Q}_\infty^{\mathsf{T}}$ since $\mathbf{A}_{n_k} \to \hat{\mathbf{B}}$. And since $\mathbf{Q}_n\mathbf{A}_n\mathbf{Q}_n^{\mathsf{T}} \to \mathbf{B}$, we also have $\mathbf{Q}_{n_k}\mathbf{A}_{n_k}\mathbf{Q}_{n_k}^{\mathsf{T}} \to \mathbf{B}$. Therefore $\mathbf{B} = \mathbf{Q}_\infty\hat{\mathbf{B}}\mathbf{Q}_\infty^{\mathsf{T}}$, and

$$\begin{aligned}
\boldsymbol{\Psi}_r(\mathbf{A}_{n_k})[\mathbf{E}_1, \ldots, \mathbf{E}_r] &= \boldsymbol{\Psi}_r(\mathbf{Q}_{n_k}\mathbf{A}_{n_k}\mathbf{Q}_{n_k}^{\mathsf{T}})[\mathbf{Q}_{n_k}\mathbf{E}_1\mathbf{Q}_{n_k}^{\mathsf{T}}, \ldots, \mathbf{Q}_{n_k}\mathbf{E}_r\mathbf{Q}_{n_k}^{\mathsf{T}}] \\
&\to \hat{\boldsymbol{\Psi}}_r(\mathbf{B})[\mathbf{Q}_\infty\mathbf{E}_1\mathbf{Q}_\infty^{\mathsf{T}}, \ldots, \mathbf{Q}_\infty\mathbf{E}_r\mathbf{Q}_\infty^{\mathsf{T}}] \\
&= \hat{\boldsymbol{\Psi}}_r(\mathbf{Q}_\infty\hat{\mathbf{B}}\mathbf{Q}_\infty^{\mathsf{T}})[\mathbf{Q}_\infty\mathbf{E}_1\mathbf{Q}_\infty^{\mathsf{T}}, \ldots, \mathbf{Q}_\infty\mathbf{E}_r\mathbf{Q}_\infty^{\mathsf{T}}] \\
&= \hat{\boldsymbol{\Psi}}_r(\hat{\mathbf{B}})([\mathbf{E}_1, \ldots, \mathbf{E}_r]),
\end{aligned}$$

where the first equality follows from the isotropy of $\boldsymbol{\Psi}_r$ on $\mathcal{U}_3$, and the last equality from the isotropy of $\hat{\boldsymbol{\Psi}}_r$ on the orbit of $\mathbf{B}$. We have shown that for every sequence $\{\mathbf{A}_n\}$ in $\mathcal{U}_3$ converging to $\hat{\mathbf{B}}$, there is a subsequence $\{\mathbf{A}_{n_k}\}$ such that $\boldsymbol{\Psi}_r(\mathbf{A}_{n_k})[\mathbf{E}_1, \ldots, \mathbf{E}_r]$ converges to $\hat{\boldsymbol{\Psi}}_r(\hat{\mathbf{B}})[\mathbf{E}_1, \ldots, \mathbf{E}_r]$. It follows that

$$\lim_{\substack{\mathbf{A}\to\hat{\mathbf{B}} \\ \mathbf{A}\in\mathcal{U}_3}} \boldsymbol{\Psi}_r(\mathbf{A})[\mathbf{E}_1, \ldots, \mathbf{E}_r] = \hat{\boldsymbol{\Psi}}_r(\hat{\mathbf{B}})[\mathbf{E}_1, \ldots, \mathbf{E}_r]$$

for any symmetric tensors $\mathbf{E}_1, \ldots, \mathbf{E}_r$. Thus $\lim_{\mathbf{A}\to\hat{\mathbf{B}}} \boldsymbol{\Psi}_r(\mathbf{A})$, $\mathbf{A} \in \mathcal{U}_3$, exists and equals $\hat{\boldsymbol{\Psi}}_r(\hat{\mathbf{B}})$. Since $\hat{\mathbf{B}}$ was an arbitrary point in an arbitrary orbit of $\mathcal{U}_1 \cup \mathcal{U}_2$, condition (2) holds. Since $\hat{\boldsymbol{\Psi}}_r$ is isotropic on each orbit of $\mathcal{U}_1 \cup \mathcal{U}_2$, it is isotropic on $\mathcal{U}_1 \cup \mathcal{U}_2$. So if we set $\hat{\boldsymbol{\Psi}}_r = \boldsymbol{\Psi}_r$ on $\mathcal{U}_3$, then $\hat{\boldsymbol{\Psi}}_r$ is a continuous isotropic extension of $\boldsymbol{\Psi}_r$ to $\mathcal{U}$. □

As discussed above, $\boldsymbol{\Psi}_r$ is just a scalar-valued isotropic function when $r = 0$, so Proposition 2.1 also follows from Proposition 6.1.[24] Here we are primarily interested in the case $r \geq 1$ with $\boldsymbol{\Psi}_r = D^r\psi$ for some scalar-valued isotropic function ψ on

[24]Note that condition (4) in Proposition 6.1 (which for $r = 0$ is obtained by omitting the $\mathbf{E}_1, \ldots, \mathbf{E}_r$) is weaker than (4) of Proposition 2.1.

$\mathcal{U}_3$. Then Proposition 6.1 yields sufficient conditions for the existence of a continuous extension of this rth-order derivative from $\mathcal{U}_3$ to $\mathcal{U}$. It is tempting to conclude that the existence of such an extension, together with the assumption that ψ is C^{r-1} on $\mathcal{U}$, implies that ψ is C^r on $\mathcal{U}$. This is not true in general, that is, it would not be true if $\mathcal{U}_3$ was just an arbitrary open dense subset of $\mathcal{U}$.[25] The conclusion is true for the case considered here because the set $\mathcal{U} - \mathcal{U}_3 = \mathcal{U}_1 \cup \mathcal{U}_2$ of all tensors in $\mathcal{U}$ with at least two equal eigenvalues is sufficiently nice. Of course, this requires some proof. Fortunately, the result we need has been established by Ball.[26]

Proposition 6.2 (Ball [13]) *Let $r \geq 1$ and let $\mathbf{\Psi}$ be a (not necessarily isotropic) C^{r-1} function from $\mathcal{U}$ into $\mathbb{R}^n$. If $\mathbf{\Psi}$ is C^r on $\mathcal{U}_3$, and if $D^r\mathbf{\Psi}$ has a continuous extension to $\mathcal{U}$, then $\mathbf{\Psi}$ is C^r on $\mathcal{U}$.*

Then Proposition 4.1 follows from Proposition 6.2 (with $\mathbf{\Psi} = \psi$, $n = 1$) and Proposition 6.1 (with $\mathbf{\Psi}_r = D^r\psi$).

[25] As pointed out by Ball [13, §2], the Cantor function provides a counterexample for $r = 1$.

[26] The result stated here follows from the much more general Proposition 2.2 in Ball's paper, together with the fact (which is proved on p. 714 of his paper) that $Sym_1 \cup Sym_2$ is closed and *sparse* (cf. his Definition 2.1).

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